Time change

Almut E. D. Veraart
CREATES
University of Aarhus
Aarhus
Denmark
+45 8942 2142
averaart@creates.au.dk

Matthias Winkel
Department of Statistics
University of Oxford
Oxford
UK
tel. +44 1865 272875
fax. +44 1865 272595
winkel@stats.ox.ac.uk

TimeChange8.tex – LaTeX2e

Abstract

The mathematical concept of time–changing continuous–time stochastic processes can be regarded as one of the standard tools for building financial models. This article reviews briefly the theory on time–changed stochastic processes and relates them to stochastic volatility models in finance. Popular models, including time–changed Lévy processes, where the time–change process is given by a subordinator or an absolutely continuous time change, are presented. Finally, we discuss the potential and the limitations of using such processes for constructing multivariate financial models.

Keywords: Time change; business time; stochastic volatility; subordinator; Lévy process; semimartingale.

The mathematical concept of time–changing continuous–time stochastic processes has first been studied in [11, 17, 18, 24, 25, 31, 41] and has later been introduced to the finance literature in [14]. Today, it can be regarded as one of the standard tools for building financial models (see also [23, 34]).

Let $X = (X_t)_{t\geq 0}$ denote a stochastic process, sometimes referred to as the *base process*, and let $T = (T_s)_{s\geq 0}$ denote a non–negative, non–decreasing stochastic process not necessarily independent of X. The *time–changed process* is then defined as $Y = (Y_s)_{s\geq 0}$, where

$$Y_s = X_{T_s}. \tag{1}$$

The process X is said to evolve in *operational time.* The process T is referred to as *time change*, stochastic clock, chronometer or *business time.* It reflects the varying *speed* of Y.

This article is structured as follows. First we link the use of time–changed stochastic processes to the construction of univariate stochastic volatility models in finance. Then we focus on the choice of an appropriate base process and, next, on popular examples for the time change. Finally, we briefly discuss the potential and the limitations of the methodology of time–changing stochastic processes to construct multivariate models in finance.

Asset price models with stochastic volatility

The use of time–changed stochastic processes in finance is closely linked to the concept of stochastic volatility models for asset prices. Numerous empirical studies have revealed the fact that asset price volatility tends to be time–varying and tends to show clustering effects. The concept of stochastic volatility (see eqf19 019) in continuous–time asset price models on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ can basically be introduced by two methods.

One of the methods is to use time–changed stochastic processes as in (1), where natural assumptions are that $X = (X_t)_{t\geq 0}$ is an $\mathcal{F}_t$–semimartingale and $(T_s)_{s\geq 0}$ is an increasing family of $\mathcal{F}_t$–stopping times. The base process X is often assumed to possess some homogeneity properties whereas a non–linear time change can induce deviations from homogeneity.

The other method is to use stochastic integrals (see eqf02 013) of the form

$$Y_t = \int_0^t \sigma_{r-} dX_r, \tag{2}$$

where $\sigma = (\sigma_t)_{t \geq 0}$ is a non–negative $\mathcal{F}_t$–predictable stochastic volatility process and $X = (X_t, t \geq 0)$ is an $\mathcal{F}_t$–semimartingale. Often, X is assumed to possess some homogeneity properties so that non–constant σ can induce deviations from homogeneity.

Under certain conditions, the models (1) and (2) lead to equivalent models. However, this is not true in general and we will highlight in the following some of the main differences between these two modelling approaches.

Time–changed Lévy processes

In the finance literature, the main focus is on time–changed Brownian motion or, more generally, on time–changed Lévy processes (see eqf02 004, eqf08 029), since these processes possess natural homogeneity properties of stationary and independent increments (returns).

Time–changed Brownian motion has first been used as a model for (logarithmic) asset prices by Clark [14]. He investigated the case where $X = B = (B_t)_{t \geq 0}$ is standard Brownian motion and where T is an independent continuous time change. Clearly, in such a setting, the time–changed process $Y_s = B_{T_s}$ has a mixed normal distribution, i.e. $Y_s | T_s \sim N(0, T_s)$, and is a continuous local martingale. The power of such models and those with the assumptions of independence and/or continuity relaxed is expressed in the following key results.

Theorem 1 (Dubins–Schwarz [18, 36, 37]) *Every continuous local martingale $M = (M_s)_{s \geq 0}$ can be written as a time–changed Brownian motion $(B_{[M]_s})_{s \geq 0}$, where $[M] = ([M]_s)_{s \geq 0}$ is the (continuous) quadratic variation of M.*

Also $M_t = \int_0^t \sigma_{r-} dW_r$ for $X = W = (W_t)_{t \geq 0}$ Brownian motion and $\sigma = (\sigma_t)_{t \geq 0}$ independent non–negative with càdlàg sample paths is a continuous local martingale with quadratic variation $[M]_t = \left[\int_0^\cdot \sigma_{r-} dW_r\right]_t = \int_0^t \sigma_r^2 dr$, which is often referred to as integrated variance.

Corollary 2 *In the context of the Dubins–Schwarz Theorem, for the local martingale $M_s = \int_0^s \sigma_{r-} dW_r = B_{[M]_s}$ independence of W and σ is equivalent to independence of B and $T = [M]$.*

Therefore, the models (1) and (2) are equivalent if X is Brownian motion and $T_s = \int_0^s \sigma_r^2 dr$ is an absolutely continuous time change. Regarding Clark's independence assumption, we note that independence of σ and X is also equivalent in (1) and (2) – however, this excludes the leverage effect, i.e. the usually negative correlation between asset returns and volatility.

Fundamentally, these results are due to the scaling property of Brownian motion X, which translates spatial scaling σX_t or (2) into temporal scaling $X_{\sigma^2 t}$ or (1). Also, if Brownian motion is replaced by an α–stable Lévy process and $T_t = \int_0^t \sigma_s^\alpha ds$, the two models (1) and (2) are basically equivalent (see [27, 28]). No other Lévy process has such a scaling property, and indeed, the models (1) and (2) will be different. If we relate higher volatility to higher market activity, then (1) suggests that markets move at a higher speed, and (2) suggests that the volumes traded are higher.

Theorem 3 (Monroe [35]) *Every (càdlàg) semimartingale $Z = (Z_s)_{s\geq 0}$ can be written as a time–changed Brownian motion $(B_{T_s})_{s\geq 0}$ for a (càdlàg) family of stopping times $(T_s)_{s\geq 0}$ on a suitably extended probability space.*

In the light of the Fundamental Theorem of Asset Pricing (see eqf04 002), this means that every arbitrage-free model can be viewed as time-changed Brownian motion. However, this result is of limited use for the construction of simple and natural parametric models.

In the finance literature, we find many asset price models where the base process is chosen to be a Lévy process other than Brownian motion (see e.g. [12, 13]). Here, we just refer to eqf08 029 for a detailed treatment of this class of models.

We also mention Lamperti's representations [29, 30] of continuous-state branching processes and of self-similar Markov processes as time–changed Lévy processes very much in the spirit of the Dubins–Schwarz Theorem. Salminen and Yor [38] use Lamperti's time change to study Dufresne's functional (cf. [20]) that arises in the computation of discounted values in certain models of continuously payable perpetuities in actuarial science.

Choice of time change

There are different methods for choosing a time change which is suitable for financial models. Two classes of such processes are particularly popular: subordinators and absolutely continuous time changes (see e.g. [1]).

In the finance literature, the terms *time change* and *subordinator* are sometimes used synonymously. However, in probability theory, the term subordinator describes a particular class of stochastic processes (as defined below) and does not include all time changes.

Subordinators

Subordinators are non-decreasing Lévy processes (see e.g. [10, 15, 39]) and hence possess stationary and independent increments. They are pure jump processes of possibly infinite activity plus a deterministic linear drift. Clearly, they have no Brownian component and are of finite variation. Important examples include simple Poisson processes, increasing compound Poisson processes, gamma processes and (tempered) stable subordinators. Note that many popular models in finance are based on time–changed Brownian motion where the time change is chosen to be a subordinator. E.g. the Variance Gamma process ([32, 33]) can be represented as Brownian motion time–changed by a gamma process, the Normal Tempered Stable process (including the Normal Inverse Gaussian process ([2, 3])) can be written as Brownian motion time–changed by a tempered stable subordinator. Brownian motion time–changed by an independent subordinator always yields a Lévy process.

Absolutely continuous time changes

Another important class of time changes is given by the class of absolutely continuous time changes of the form $T_s = \int_0^s \tau_u du$, for a positive and integrable process $\tau = (\tau_s)_{s\geq 0}$. Note that in such a setting T is always continuous, but τ can exhibit jumps. The process τ is often called instantaneous (business) activity rate. Such models have been studied in the context when X is a Lévy process by Carr et al. [12] and [13] amongst others. The advantage of this model class is that it leads to affine models which are highly analytically tractable (see e.g. [19, 27]), whereas stochastic integrals with respect to Lévy processes are in general not affine. Popular examples for the instantaneous activity rate are given by the Cox–Ingersoll–Ross process and the non–Gaussian Ornstein-Uhlenbeck process (see [7, 8, 16]).

Multivariate setting

Time–changed stochastic processes are also used in the finance literature to construct multivariate models. Usually, the base process X is then assumed to be multivariate (e.g. a multivariate Brownian motion) and the time change process T is still assumed to be univariate (see e.g. [15, 21] and the references therein). The advantage of such a modelling framework is the fact that such processes are highly analytically tractable and easy to simulate from. However, the range of dependence between the univariate components in such a multivariate model is rather limited (and in particular does not even include complete independence). Furthermore, such a model does not allow for an arbitrary choice of univariate models for the components (see [15]). The mathematical properties of multivariate processes X time–changed by a multivariate vector of time changes T have recently been studied in [5], but this has so far not been considered as a standard tool for constructing multivariate models in finance. So far, it has been much more common to use multivariate extensions of a stochastic integral (2) to construct multivariate stochastic volatility models, than to use time–changed processes. Finally, note that the concept of subordinators can also be generalised to matrix subordinators (see e.g. [4, 6]) and their applicability in financial models is subject to ongoing research (see e.g. [9, 40] for some first references).

Acknowledgement

We would like to thank Marc Yor and Ole Barndorff-Nielsen for suggesting relevant references. The first author acknowledges financial support by the Center for Research in Econometric Analysis of Time Series, CREATES, funded by the Danish National Research Foundation.

Related articles

eqf02 004, eqf02 013, eqf02 023, eqf08 022, eqf08 026, eqf08 029, eqf19 019, eqf19 020, eqf19 021.

References

[1] Ané, T. and Geman, H. [2000], 'Order flow, transaction clock, and normality of asset returns', *The Journal of Finance* **55**(5), 2259–2284.

[2] Barndorff-Nielsen, O. E. [1997], 'Normal inverse gaussian distributions and stochastic volatility modelling', *Scandinavian Journal of Statistics* **24**, 1– 13.

[3] Barndorff-Nielsen, O. E. [1998], 'Processes of normal inverse Gaussian type', *Finance and Stochastics* **2**, 41–68.

[4] Barndorff-Nielsen, O. E., Maejima, M. and Sato, K.-I. [2006], 'Some classes of multivariate infinitely divisible distributions admitting stochastic integral representation', *Bernoulli* **12**, 1–33.

[5] Barndorff-Nielsen, O. E., Pedersen, J. and Sato, K.-I. [2001], 'Multivariate subordination, selfdecomposability and stability', *Advances of Applied Probability* **33**, 160–187.

[6] Barndorff-Nielsen, O. E. and Pérez-Abreu, V. [2008], 'Matrix subordinators and related upsilon transformations', *Theory of Probability and its Applications* **52**(1), 1–23.

[7] Barndorff-Nielsen, O. E. and Shephard, N. [2001], 'Non–Gaussian Ornstein–Uhlenbeck–based models and some of their uses in financial economics (with discussion)', *Journal of the Royal Statistical Society Series B* **63**, 167–241.

[8] Barndorff-Nielsen, O. E. and Shephard, N. [2002], 'Econometric analysis of realised volatility and its use in estimating stochastic volatility models', *Journal of the Royal Statistical Society* **B 64**, 253–280.

[9] Barndorff-Nielsen, O. E. and Stelzer, R. [2007], 'Positive–definite matrix processes of finite variation', *Probability and Mathematical Statistics* **27**(1), 3–43.

[10] Bertoin, J. [1996], *Lévy processes*, Cambridge University Press, Cambridge.

[11] Bochner, S. [1949], 'Diffusion equation and stochastic processes', *Proceedings of the National Academy of Science of the United States of America* **85**, 369–370.

[12] Carr, P., Geman, H., Madan, D. B. and Yor, M. [2003], 'Stochastic volatility for Lévy processes', *Mathematical Finance* **13**(3), 345–382.

[13] Carr, P. and Wu, L. [2004], 'Time–changed Lévy processes and option pricing', *Journal of Financial Economics* **71**, 113–141.

[14] Clark, P. K. [1973], 'A subordinated stochastic process model with fixed variance for speculative prices', *Econometrica* **41**, 135–156.

[15] Cont, R. and Tankov, P. [2004], *Financial Modelling With Jump Processes*, Financial Mathematics Series, Chapman & Hall, Boca Raton, Florida, USA.

[16] Cox, J. C., Ingersoll, J. E. and Ross, S. A. [1985], 'An intertemporal general equilibrium model of asset prices', *Econometrica* **53**, 363– 384.

[17] Doeblin, W. [2000], 'Sur l'équation de Kolmogoroff', *Les Comptes Rendus de l'Académie des Sciences* **I**(331), 1031–1187. Pli cacheté déposé le 26 février 1940, ouvert le 18 mai 2000.

[18] Dubins, L. and Schwarz, G. [1965], 'On continuous martingales', *Proceedings of the National Academy of Sciences of the United States of America* **53**(5), 913–916.

[19] Duffie, D., Filipovic, D. and Schachermayer, W. [2003], 'Affine processes and applications in finance', *Annals of Applied Probability* **13**, 984–1053.

[20] Dufresne, D. [1990], 'The distribution of a perpetuity, with applications to risk theory and pension funding', *Scandinavian Actuarial Journal* pp. 39–79.

[21] Eberlein, E. [2001], Application of generalized hyperbolic Lévy motion to finance, *in* O. E. Barndorff-Nielsen, T. Mikosch and S. Resnick, eds, 'Lévy processes: Theory and Applications', Birkhäuser, Basel, pp. 319–337.

[22] Fu, M., Jarrow, R., Yen, J.-Y. and Elliott, R., eds [2007], *Advances in Mathematical Finance*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston.

[23] Geman, H. [2005], 'From measure changes to time changes in asset pricing', *Journal of Banking and Finance* **29**, 2701–2722.

[24] Hunt, G. [1957], 'Markov processes and potentials I, II and III', *Illinois Journal of Mathematics* . **1** (1957), 44–93; 316–396; **2** (1958), 151–213.

[25] Itô, K. and McKean, H. P. [1965], *Diffusion Processes and Their Sample Paths*, Springer–Verlag, Berlin.

[26] Jeanblanc, M., Yor, M. and Chesney, M. [2009], *Mathematical Methods for Financial Markets*, Springer Finance, Berlin.

[27] Kallsen, J. [2006], A didactic note on affine stochastic volatility models, *in* Y. Kabanov, R. Liptser and J. Stoyanov, eds, 'From Stochastic Calculus to Mathematical Finance', Springer, Berlin, pp. 343–368.

[28] Kallsen, J. and Shiryaev, A. [2002], 'Time change representation of stochastic integrals', *Theory of Probability and Its Applications* **46**, 522–528.

[29] Lamperti, J. [1967], 'Continuous–state branching processes', *Bulletin of the American Mathematical Society* **73**, 382–386.

[30] Lamperti, J. [1972], Semi–Stable Markov Processes. I, Vol. 22 of *Probability Theory and Related Fields*, Springer–Verlag, pp. 205– 225.

[31] Lévy, P. [1948], *Processus stochastiques et mouvement brownien*, Gauthier–Villars, Paris.

[32] Madan, D. B. and Seneta, E. [1990], 'The VG model for share market returns', *Journal of Business* **63**, 511–524.

[33] Madan, D., Carr, P. and Chang, E. [1998], 'The variance gamma process and option pricing', *European Finance Review* **2**, 79– 105.

[34] McKean, H. [2002], Scale and clock, *in* H. Geman, D. Madan, S. Pliska and T. Vorst, eds, 'Mathematical Finance – Bachelier Congress 2000', Springer Finance, Berlin.

[35] Monroe, I. [1978], 'Processes that can be embedded in Brownian motion', *The Annals of Probability* **6**(1), 42–56.

[36] Protter, P. E. [2004], *Stochastic Integration and Differential Equations*, second edn, Springer, London.

[37] Revuz, D. and Yor, M. [2001], *Continuous Martingales and Brownian Motion*, third edn, Springer, Berlin.

[38] Salminen, P. and Yor, M. [2005], 'Perpetual Integral Functionals as Hitting and Occupation Times', *Electronic Journal of Probability* **10**, 371–419.

[39] Sato, K. [1999], *Lévy processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge.

[40] Stelzer, R. J. [2007], Multivariate Continuous time stochastic volatility models driven by a Lévy process, Dissertation, Technische Universität München,

München. Urn: http://nbn-resolving.de/urn/resolver.pl?urn:nbn:de:bvb:91-diss-20070704-624065-1-3.

[41] Volkonski, V. [1958], 'Random time changes in strong Markov processes', *Theory of Probability and its Applications* **3**, 310–326.