o13
Actuarial Science

Matthias Winkel[1]
University of Oxford

MT 2003

[1]Departmental lecturer at the Department of Statistics, supported by Aon Ltd. and the Institute of Actuaries

o13
Actuarial Science

16 lectures MT 2003 and 16 lectures HT 2004

Aims

This course is supported by the Institute of Actuaries. It is designed to give the undergraduate mathematician an introduction to the financial and insurance worlds in which the practising actuary works. Students will cover the basic concepts of risk management models for mortality and sickness, and for discounted cash flows. In the final examination, a student obtaining at least an upper second class mark on paper o13 can expect to gain exemption from the Institute of Actuaries' paper 102, which is a compulsory paper in their cycle of professional actuarial examinations.

Synopsis

Fundamental nature of actuarial work. Use of generalised cash flow model to describe financial transactions. Time value of money using the concepts of compound interest and discounting. Present values and the accumulated values of a stream of equal or unequal payments using specified rates of interest and the net present value at a real rate of interest, assuming a constant rate of inflation. Interest rates and discount rates in terms of different time periods. Compound interest functions, equation of value, loan repayment, project appraisal. Investment and risk characteristics of investments. Simple compound interest problems. Price and value of forward contracts. Term structure of interest rates, simple stochastic interest rate models. Single decrement model, present values and the accumulated values of a stream of payments taking into account the probability of the payments being made according to a single decrement model. Annuity functions and assurance functions for a single decrement model. Liabilities under a simple assurance contract or annuity contract.

Reading

All of the following are available from the Publications Unit, Institute of Actuaries, 4 Worcester Street, Oxford OX1 2AW

- Subject 102: Financial Mathematics. Core reading 2003. Faculty & Institute of Actuaries 2002
- J J McCutcheon and W F Scott, An Introduction to the Mathematics of Finance, Heinemann 1986
- P Zima and R P Brown, Mathematics of Finance, McGraw-Hill Ryerson 1993
- H U Gerber, Life Insurance Mathematics, Springer 1990
- N L Bowers et al, Actuarial mathematics, 2nd edition, Society of Actuaries 1997

Contents

Lecture 1

Introduction

This introduction is two-fold. First, we give some general indications on the work of an actuary. Second, we introduce cash flow models as the basis of this course and a suitable means to describe and look beyond the contents of this course.

1.1 The actuarial profession

Actuarial Science is an old discipline. The Institute of Actuaries was formed in 1848, (the Faculty of Actuaries in Scotland in 1856), but the profession is much older. An important root is the construction of the first life table by Sir Edmund Halley in 1693. However, this does not mean that Actuarial Science is oldfashioned. The language of probability theory was gradually adopted between the 1940s and 1970s. The development of the computer has been reflected and exploited since its early days. The growing importance and complexity of financial markets currently changes the profession.

Essentially, the job of an actuary is risk assessment. Traditionally, this was insurance risk, life insurance and later general insurance (health, home, property etc). As typically enourmous amounts of money, reserves, have to be maintained, this naturally extended to investment strategies including the assessment of risk in financial markets. Today, the Faculty and Institute of Actuaries claim in their slogan yet more broadly to make "financial sense of the future".

To become an actuary in the UK, one has to pass nine mathematical, statistical, economic and financial examinations (100 series), an examination on communication skills (201), an examination in each of the five specialisation disciplines (300 series) and for a UK fellowship an examination on UK specifics of one of the five specialisation disciplines. This whole programme takes normally at least three or four years after a mathematical university degree and while working for an insurance company.

This course is an introductory course where important foundations are laid and an overview of further actuarial education and practice is given. The 101 paper is covered by the second year probability and statistics course. An upper second mark in the examination following this course normally entitles to an exemption from the 102 paper. Two thirds of the course concern 102, but we also touch upon material of 103, 104, 105 and 109.

1.2 The generalised cash flow model

The cash flow model systematically captures cash payments either between different parties or, as we shall focus on, in an in/out way from the perspective of one party. This can be done at different levels of detail, depending on the purpose of an investigation, the complexity of the situation, the availability of reliable data etc.

Example 1 Look at the transactions on a worker's monthly bank statement

Date	Description	Money out	Money in
01-09-02	Gas-Elec-Bill	£21.37	
04-09-02	Withdrawal	£100.00	
15-09-02	Telephone-Bill	£14.72	
16-09-02	Mortgage Payment	£396.12	
28-09-02	Withdrawal	£150.00	
30-09-02	Salary		£1022.54

Extracting the mathematical structure of this example we define elementary cash flows.

Definition 1 A *cash flow* is a vector $(t_j, c_j)_{1\leq j\leq m}$ of *times* $t_j \geq 0$ and *amounts* $c_j \in \mathbb{R}$. Positive amounts $c_j > 0$ are called *inflow*. If $c_j < 0$, then $-c_j$ is called *outflow*.

Example 2 The cash flow of Example 1 is mathematically given by

j	t_j	c_j
1	1	-21.37
2	4	-100.00
3	15	-14.72

j	t_j	c_j
4	16	-396.12
5	28	-150.00
6	30	1022.54

Often, the situation is not as clear as this, and there may be uncertainty about the time/amount of a payment. This can be modelled using probability theory.

Definition 2 A *generalised cash flow* is a *random* vector $(T_j, C_j)_{1\leq j\leq M}$ of *times* $T_j \geq 0$ and *amounts* $C_j \in \mathbb{R}$ with a possibly random length $M \in \mathbb{N}$.

Sometimes, in fact always in this course, the random structure is simple and the times or the amounts are deterministic, or even the only randomness is that a well specified payment may fail to happen with a certain probability.

Example 3 Future transactions on a worker's bank account

j	T_j	C_j	Description
1	1	-21.37	Gas-Elec-Bill
2	T_2	C_2	Withdrawal?
3	15	C_3	Telephone-Bill

j	T_j	C_j	Description
4	16	-396.12	Mortgage payment
5	T_5	C_5	Withdrawal?
6	30	1022.54	Salary

Here we assume a fixed Gas-Elec-Bill but a varying telephone bill. Mortgage payment and salary are certain. Any withdrawals may take place. For a full specification of the generalised cash flow we would have to give the (joint!) laws of the random variables.

This example shows that simple situations are not always easy to model. It is an important part of an actuary's work to simplify reality into tractable models. Sometimes, it is worth dropping or generalising the time specification and just list approximate or qualitative ('big', 'small', etc.) amounts of income and outgo. Cash flows can be represented in various ways as the following more relevant examples illustrate.

1.3 Actuarial science as examples in the generalised cash flow model

Example 4 (Zero-coupon bond) Usually short term investments with interest paid at the end of the term, e.g. invest £1000 for ninety days for a return of £1010.

j	t_j	c_j
1	0	-1000
2	90	+1010

Example 5 (Government bonds, fixed interest securities) Usually long term investments with annual or semi-annual *coupon payments* (interest), e.g. invest £10000 for ten years at 5% p.a. The government borrows money from investors.

−£10000	+£500	+£500	+£500		+£500	+£10500
0	1	2	3		9	10

Alternatively, interest and redemption value may be tracking an *inflation index*.

Example 6 (Corporate bonds) They work the same as government bonds, but they are not as secure. *Rating of companies* gives an indication of security. If companies go bankrupt, invested money is often lost. One may therefore wish to add probabilities to the positive cash flows in the above figure. Typically, the interest rate in corporate bonds is higher to allow for this extra *risk of default* that the investor takes.

Example 7 (Equities) Shares in the ownership of a company that entitle to regular *dividend payments* of amounts depending on the profit of the company and decisions at its Annual General Meeting of Shareholders. Equities can be bought and sold (through a stockbroker) on *stock markets* at fluctuating market prices. In the above figure (including default probabilities) the inflow amounts are not fixed, the term at the discretion of the shareholder and the final repayment value is not fixed. There are *advanced stochastic models* for stock price evolution. A wealth of derivative products is also available, e.g. forward contracts, options to sell or buy shares, also funds to spread risk.

Example 8 (Annuity-certain) Long term investments that provide a series of regular annual (semi-annual or monthly) payments for an initial lump sum, e.g.

−£10000	+£1400	+£1400	+£1400		+£1400	+£1400
0	1	2	3		9	10

Example 9 (Interest-only loan) Formally in the cash flow model the inversion of a bond, but the rights of the parties are not exactly inverted. Whereas the bond investor can usually redeem early with only minor restrictions, the lender of a loan normally has to obey stricter rules, for the benefit of the borrower.

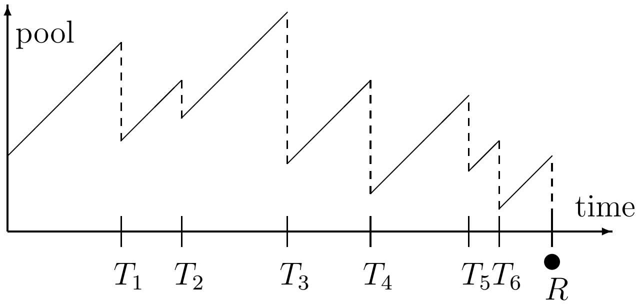
Example 10 (Repayment loan) Formally in the cash flow model the inversion of an annuity-certain, but with differences in the rights of the parties as for interest-only loans.

Example 11 (Life annuity) The only difference to Annuity-certain is the term of payments. Instead of having a fixed term life annuities terminate on death of the holder. Risk components like his age, health and profession when entering the contract determine the amount of the initial deposit. They are usually issued by insurance companies. Several modifications exist (minimal term, maximal term, payable from the death of one person to a second person for their life etc.).

Example 12 (Term assurance) They pay a lump sum on death (or serious illness) for monthly or annual premiums that depend on age and health of the policy holder when the policy is underwritten. A typical assurance period is twenty years, but age limits of sixty-five or seventy years are common. The amount can be reducing in accordance with an outstanding mortgage. There is no cash in value at any time.

Example 13 (Endowment assurance) They have the same conditions as term assurances but also offer payment in the event of survival of the term. Due to this they are much more expensive. They increase in value and can be sold early if needed.

Example 14 (Property insurance) They are one class of general insurance (others are health, building, motor etc.). For regular premiums the insurance company replaces or refunds any stolen or damaged objects included in the policy. From the provider's point of view, all policy holders pay into a pool for those that have claims. Claim history of policy holders affects their premium.



A branch of an insurance company is said to suffer *technical ruin* if the pool runs empty.

Example 15 (Appraisal of investment projects) E.g., consider the investment into a building project. An initial construction period requires certain negative cash flows, the following exploitation (e.g. letting) essentially positive cash flows, but maintenance has to be taken into accout as well. Under what circumstances is the project profitable?

More generally, one can assess whole companies on their profitability. The important and difficult first step is estimating the in- and outflows. This should be done by an independent observer to avoid manipulation. It is common practice to compare average, optimistic and pessimistic estimations reflecting an implicit underlying stochastic model.

Lecture 2

The theory of compound interest

Quite a few problems addressed and solved in this course can be approached in an intuitive way. However, it adds to clarity and understanding to specify a mathematical model in which the concepts and methods can be discussed. The concept of cash flows seen in the last lecture is one part of this model. In this lecture, we shall construct the basic compound interest model in which interest of capital investments under varying interest rates can be computed. This model will play a role during the whole course, with suitable extensions from time to time.

Whenever mathematical models are used, reality is only partially represented. Important parts of mathematical modelling are the discussions of model assumptions and parameter specification, particularly the interpretation of model results in reality. It is instructive to read these lecture notes with this in mind.

2.1 Simple versus compound interest

Consider an investment of C for t time units at the end of which $S = C + I$ is returned. Then we call t the *term*, I the *interest* and S the *accumulated value* of the *initial capital* C. One might want to call $i = I/tC$ the *interest rate* per unit time, but there are different types of interest rates that need to be distinguished, so we have to be more precise.

Definition 3 *$I = I_{simp}(i, t, C) = tiC$ is called* simple interest *on the initial capital $C \in \mathbb{R}$ invested for $t \in \mathbb{R}_+$ time units at the* interest rate *$i \in \mathbb{R}_+$ per unit time.*

$$S = S_{simp}(i, t, C) = C + I_{simp}(i, t, C) = (1 + ti)C$$

is called the accumulated value of C after time t under simple interest at rate i.

Interest rates always refer to some time unit. The standard choice is one year, but it sometimes eases calculations to choose one month or one day. The definition reflects the assumption that the interest rate does not vary with the initial capital nor the term.

The problem with simple interest is that splitting the term $t = t_1 + t_2$ and reinvesting the accumulated value after time t_1 yields

$$S_{simp}(i, t_2, S_{simp}(i, t_1, C)) = (1 + ti + t_1 t_2 i^2)C > (1 + ti)C = S_{simp}(i, t, C),$$

provided only $0 < t_1 < t$ and $i > 0$. This profit by term splitting has the disadvantageous effect that the customer who maximises his profit keeps reinvesting his capital for short periods to achieve interest on his interest, so-called compound interest. In fact, he would have to choose infinitesimally small periods:

Proposition 1 *For any given interest rate $\delta \in \mathbb{R}_+$ we have*

$$\begin{aligned}\sup_{n\in\mathbb{N},t_1,\ldots,t_n\in\mathbb{R}_+:t_1+\ldots+t_n=t} & S_{simp}(\delta, t_n, \ldots S_{simp}(\delta, t_2, S_{simp}(\delta, t_1, C))\ldots) \\ &= \lim_{n\to\infty} S_{simp}(\delta, t/n, \ldots S_{simp}(\delta, t/n, S_{simp}(\delta, t/n, C))\ldots) \\ &= e^{t\delta}C\end{aligned}$$

for all $C \in \mathbb{R}$ and $t \in \mathbb{R}_+$.

Proof: For the second equality we first establish

$$S_{simp}(\delta, t/n, \ldots S_{simp}(\delta, t/n, S_{simp}(\delta, t/n, C))\ldots) = \left(1 + \frac{t\delta}{n}\right)^n C$$

by induction from the definition of S_{simp}. Then we use the continuity and power expansion of the natural logarithm to see the existence of the limit and

$$\log\left(\lim_{n\to\infty}\left(1+\frac{t\delta}{n}\right)^n\right) = \lim_{n\to\infty} n\log\left(1+\frac{t\delta}{n}\right) = \lim_{n\to\infty} n\left(\frac{t\delta}{n} + O(1/n^2)\right) = t\delta.$$

Furthermore, the first equality follows from the observation

$$1 + t\delta \le \sum_{k\ge0}\frac{(t\delta)^k}{k!} = e^{t\delta} \quad\Rightarrow\quad (1+t_1\delta)(1+t_2\delta)\ldots(1+t_n\delta) \le e^{t_1\delta+\ldots+t_n\delta} = e^{t\delta}$$

for all $t_1 \ldots t_n \in \mathbb{R}_+$ with $t_1 + \ldots + t_n = t$, and this inequality is preserved when we take the supremum over all such choices of $t_1, \ldots, t_n$. □

We changed our notation for the interest rate from i to δ since this compounding of interest allows different quantities to be called interest rate. In particular, if we apply the "optimal strategy" of Proposition 1 to an initial capital of 1, the return after one time unit is $e^\delta = 1 + (e^\delta - 1)$ and $i = e^\delta - 1$ is a natural candidate to be called the interest rate per unit time. Note that then

$$e^{t\delta} = (1+i)^t \quad\text{and}\quad \delta = \frac{\partial}{\partial t}(1+i)^t\bigg|_{t=0}.$$

Definition 4 *$S = S_{comp}(i, t, C) = (1+i)^t C$ is called the* accumulated value *of $C \in \mathbb{R}$ after $t \in \mathbb{R}_+$ time units under* compound interest *at the* effective interet rate *$i \in \mathbb{R}_+$ per unit time. $I = I_{comp}(i, t, C) = S_{comp}(i, t, C) - C$ is called* compound interest *on C after n time units at rate i per unit time.*

$$\delta = \log(1+i) = \frac{\partial}{\partial t} I_{comp}(i, t, 1)\bigg|_{t=0}$$

is called the force of interest*.*

Compound interest is the standard for long term investments. One might say, simple interest is oldfashioned, but it is still used for short term investments when the difference to compound interest is relatively small. Interest cannot be paid continuously even if the tendency is to increase the frequency of interest payments (used to be annually, now quaterly or even monthly). Within one such time unit, any interest is usually calculated as simple interest and credited at the end of each time unit. Of course one could also credit compound interest at the end of time units only. However, for the investor, the use of simple interest is an advantage, since

Proposition 2 *Given an effective interest rate $i > 0$ and an initial capital $C > 0$,*

$$\begin{array}{lcl} 0 < t < 1 & \Rightarrow & I_{simp}(i,t,C) > I_{comp}(i,t,C) \\ 1 < t < \infty & \Rightarrow & I_{simp}(i,t,C) < I_{comp}(i,t,C) \end{array}$$

Proof: We compare accumulated values. The strict convexity of $f(t) = (1+i)^t$ follows by differentiation. But then we have for $0 < t < 1$

$$f(t) < tf(1) + (1-t)f(0) = t(1+i) + (1-t) = 1 + ti = g(t)$$

and for $t > 1$

$$f(1) < \frac{1}{t}f(t) + \left(1 - \frac{1}{t}\right)f(0) \quad \Rightarrow \quad f(t) > tf(1) + (1-t)f(0) = g(t).$$

This completes the proof since $S_{simp}(i,t,C) = Cg(t)$ and $S_{comp}(i,t,C) = Cf(t)$. □

Rates quoted by banks are not always effective rates. Therefore, comparison of different types of interest should be made with care. This statement will be supported for instance by the discussion of nominal interest rates in the next lecture.

As indicated earlier, we shall relax our heavy notation in the sequel, e.g. $I_{comp}(i,t,C)$ to (I_{comp} or) I, whenever there is no ambiguity. Simple interest will hardly play a role in what follows, but compound interest is the basis of the remainder of this course.

Example 16 Given an interest rate of $i = 4\%$ per annum (p.a.). Investing $C = £1000$ for $t = 5$ years yields

$$I_{simp} = tiC = £200.00 \qquad \text{and} \qquad I_{comp} = \left((1+i)^t - 1\right)C = £216.65$$

2.2 Time-dependent interest rates

In the previous section we assumed that interest rates are constant over time. Suppose, we now let $i = i(t)$ vary with discrete time $t \in \mathbb{N}_+$. If we want to avoid odd effects by term splitting, we should define the accumulated value at time n for an investment of C at time 0 as

$$(1 + i(n-1)) \dots (1 + i(1)) \dots (1 + i(0))C.$$

When passing from integer terms to non-integer terms, it turns out that instead of specifying i, we had better specify the force of interest $\delta(t)$ which we saw to have a local meaning as the derivative of the compound interest function.

More explicitly, if $\delta(\cdot)$ is piecewise constant, then the iteration of S (S_{comp} now and forever) along the successive subterms $t_j \in \mathbb{R}_+$ at constant forces of interest δ_j yields the return of an initial amount C

$$S = e^{\delta_n t_n} \dots e^{\delta_2 t_2} e^{\delta_1 t_1} C \tag{1}$$

and we can see this as the exponential Riemann sum defining $\exp(\int_0^t \delta(s)ds)$.

Definition 5 Given a time dependent force of interest $\delta(t)$, $t \in \mathbb{R}_+$, that is (locally) Riemann integrable, we define the *accumulated value* at time $t \geq 0$ of an initial capital $C \in \mathbb{R}$ *under a force of interest* δ as

$$S = C \exp\left\{ \int_0^t \delta(s)ds \right\}.$$

$I = S - C$ is called the *interest of* C *for time* t *under* δ.

$\delta(t)$ can be seen as defining the environment in which the value of invested capital evolves. We will see in the next lecture that this definition provides the most general (deterministic) setting, under some weak regularity conditions and under a consistency condition (consistency under term splitting), in which we can attribute time values to cash flows $(t_j, c_j)_{j=1,\dots,m}$.

Local Riemann integrability is a natural assumption that makes the expressions meaningful. In fact, for practical use, only (piecewise) continuous functions δ are of importance, and there is no reason for us go beyond this.

In support of this definition, we conclude by quoting some results from elementary calculus and Riemann integration theory. They show ways to see our definition as the only continuous (in δ!) extension of the natural definition (1) for piecewise continuous force of interest functions.

Lemma 1 *Every continuous function* $f : [0, \infty) \to \mathbb{R}$ *can be approximated locally uniformly by piecewise constant functions* f_n.

This naturally extends to functions that are piecewise continuous with left and right limits on the discrete set of discontinuities. Furthermore, without any continuity assumptions, we have the convergence of integrals:

Lemma 2 *If* $f_n \to f$ *locally uniformly for Riemann integrable functions* f_n, $n \in \mathbb{N}$, *then*

$$\int_0^t f_n(s)ds \to \int_0^t f(s)ds$$

locally uniformly as a function of $t \geq 0$.

Also, although the approximation by upper and lower Riemann sums is not locally uniform, in general, the definition of Riemann integrability forces the convergence of the Riemann sums (which are integrals of approximations by piecewise constant functions).

Lecture 3

The valuation of cash flows

This lecture combines the concepts of the first two lectures, cash flows and the compound interest model by valuing the former in the latter. We also introduce and value continuous cash flows.

3.1 Accumulation factors and consistency

In the previous lecture we defined an environment for the evolution of the accumulated value of capital investments via a time-dependent force of interest $\delta(t)$, $t \in \mathbb{R}_+$. The central formula gives the value at time $t \geq 0$ of an initial investment of $C \in \mathbb{R}$ at time 0:

$$S(0,t) = C \exp\left\{\int_0^t \delta(s)ds\right\} =: C\ A(0,t) \tag{1}$$

where we call $A(0,t)$ the *accumulation factor* from 0 to t. It is the factor by which capital invested at time 0 increases until time t. We also introduce $A(s,t)$ as the factor by which capital invested at time s increases until time t. The representation in terms of δ is intuitively obvious, but we can also derive this from the following important term splitting *consistency assumption*

$$A(r,s)A(s,t) = A(r,t) \quad \text{for all } t \geq s \geq r \geq 0. \tag{2}$$

Proposition 3 *Under definition (1) and the consistency assumption (2) we have*

$$A(s,t) = \exp\left\{\int_s^t \delta(r)dr\right\} \quad \textit{for all } t \geq s \geq 0. \tag{3}$$

The accumulated value at time s of a cash flow $c = (t_j, c_j)_{1\leq j\leq m}$, ($t_j \leq s$, $j = 1, \ldots, m$) is given by

$$AVal_s(c) := \sum_{j=1}^m A(t_j, s)c_j. \tag{4}$$

Proof: For the first statement choose $r = 0$ in (2), apply (1) and solve for $A(s,t)$.

By definition of $A(s,t)$, any investment of c_j at time t_j increases to $A(t_j,s)c_j$ by time s. The second statement now follows adding up this formula over $j = 1, \ldots, m$. □

Accumulation factors are useful since they allow to move away from the reference time 0. Mathematically, this is not a big insight, but as a concept and notationally, it helps to value and nicely represent more complex structures.

We can strengthen the first part of Proposition 3 considerably as follows.

Proposition 4 *Suppose, $A : [0, \infty)^2 \to (0, \infty)$ satisfies the consistency assumption (2) and is continuously differentiable in the second argument for every fixed first argument. Then there exists a continuous function δ such that (3) holds.*

Proof: First note that the consistency assumption for $r = s = t$ implies $A(t,t) = 1$ for all $t \geq 0$. Then define

$$\delta(t) := \lim_{h \downarrow 0} \frac{A(t, t+h) - 1}{h} = \lim_{h \downarrow 0} \frac{A(0, t+h) - A(0,t)}{hA(0,t)},$$

the second equality by the consistency assumption. Now define $g(t) = A(0,t)$, $f(t) = \log(A(0,t))$, then we have

$$\delta(t) = \frac{g'(t)}{g(t)} = f'(t) \quad \Rightarrow \quad \log(A(0,t)) = f(t) = \int_0^t \delta(s) ds$$

which is (1) and by the preceding proposition, the proof is complete. □

This result shows that the concrete and elementary consistency assumption naturally leads to our models specified in a more abstract way by a time-dependent force of interest.

Corollary 1 *For any (locally) Riemann integrable δ, the accumulated value $h(t) = S(0,t)$ is the unique (continuous) solution to*

$$h'(t) = \delta(t) h(t), \qquad h(0) = C.$$

Proof: This can be seen as in the preceding proof, since $h(t) = Cg(t)$. □

3.2 Discounting and the time value of money

So far our presentation has been oriented towards calculating returns for investments. We now want to realise a certain return at a specified time t, how much do we have to invest today? The calculation of such present values of future returns is called discounting, and the inversion of (1) yields a *(discounted) present value* of

$$C = \frac{S}{A(0,t)} = S \exp\left\{ -\int_0^t \delta(s) ds \right\} =: S\ V(0,t) =: S\ v(t) \tag{5}$$

for a return S at time $t \geq 0$, and more generally a *discounted value* at time $s \leq t$ of

$$C_s = \frac{S}{A(s,t)} = S \exp\left\{ -\int_s^t \delta(r) dr \right\} =: S\ V(s,t) = S\ \frac{v(t)}{v(s)}. \tag{6}$$

$V(s,t)$ is called the *discount factor* from t to s, $v(t)$ the *(discounted) present value of* 1.

Proposition 5 *The discounted value at time s of a cash flow $c = (t_j, c_j)_{1\le j\le m}$, ($t_j \ge s$, $j = 1, \dots, m$) is given by*

$$DVal_s(c) = \sum_{j=1}^{m} c_j V(s, t_j) = \frac{1}{v(s)} \sum_{j=1}^{m} c_j v(t_j). \tag{7}$$

Proof: This follows adding up (6) over all in- and outflows (t_j, c_j), $j = 1, \dots, m$. □

The restriction to $t_j \ge s$ is mathematically not necessary, but eases interpretation. Our question was how much money we have to put aside today to be able to make future payments. If some of the payments happened in the past, particularly inflows, it is essential that the money remained in the system to earn the appropriate interest. But then we have the accumulated value of these past payments given by Definition 5.

Definition 6 The *time-t value* of a cash flow c is denoted by

$$Val_t(c) = AVal_t(c_{[0,t]}) + DVal_t(c_{(t,\infty)}).$$

This simple formula (with (4) and (7)) is central in investment and project appraisal that we discuss later in the course.

We conclude this section by a corollary to Propositions 3 and 5.

Corollary 2 *For all $s \le t$ we have $Val_t(c) = Val_s(c)A(s,t) = Val_s(c)\dfrac{v(s)}{v(t)}$.*

3.3 Continuous cash flows

When many small inflows (or outflows) accumulate regularly spread over time, it is practically useful and mathematically natural to consider a continuous approximation, continuous cash flows. We have seen this in Example 14 when a premium pool was assumed to increase continuously by regular premium payments.

Definition 7 A *continuous cash flow* is a (locally) Riemann integrable function $c : \mathbb{R}_+ \to \mathbb{R}$. $c(t)$ is also called the *payment rate at time t*.

The interpretation is that the total payment between s and t is $\int_s^t c(r)dr$, although this is ignoring the time value of money.

Proposition 6 *Under a force of interest $\delta(\cdot)$, a continuous cash flow c up to time t produces an accumulated value of*

$$AVal_t(c) = \int_0^t A(r,t)c(r)dr = \int_0^t \exp\left\{\int_r^t \delta(s)ds\right\} c(r)dr.$$

The discounted value at time t of the post-t cash flow c is

$$DVal_t(c) = \int_t^T V(t,r)c(r)dr = \int_t^T \exp\left\{-\int_t^r \delta(s)ds\right\} c(r)dr$$

where $T = \sup supp(c)$ can be infinite provided the limit exists.

We call $Val_t(c) = AVal_t(c_{[0,t]}) + DVal_t(c_{(t,\infty)})$ the value of c at time t. *Corollary 2 still holds, also for mixtures of discrete and continuous cash flows.*

Proof: Define $h(t) = S_t$. First we note that $h(0) = 0$ and by Corollary 1

$$h(t) = \int_0^t h(s)\delta(s)ds + \int_0^t c(s)ds.$$

Then, by differentiation $h'(t) = h(t)\delta(t) + c(t)$. Now h is the accumulated value, let's look at the discounted time 0 values

$$\eta(t) = S_t V(0,t) = h(t) \exp\left\{-\int_0^t \delta(s)ds\right\}$$

which satisfies

$$\eta'(t) = h'(t)V(0,t) - h(t)\delta(t)V(0,t) = c(t)V(0,t)$$

and integrating this yields η and then h as required.

For the discounted value at time u note that the case $u = 0$ and c supported by $[0,t]$ is given by $\eta(t) = C_0$. The general statement is obtained letting t tend to infinity and multiplying by accumulation factors $A(0,u)$ to pass from time 0 to time u. □

3.4 Constant discount rates

Let us turn to the special case of a constant force of interest $\delta \in \mathbb{R}_+$. We have seen a description of the model in terms of the *effective* interest rate $i = e^\delta - 1 \in \mathbb{R}_+$. The interpretation of i is that an investment of 1 earns interest i in one time unit. The concept of discounting can be approached in the same way.

Proposition 7 *In a model with constant force of interest δ, an investment of $v := 1 - d := e^{-\delta}$ yields a return of 1 after one time unit.*

Proof: Just apply (5), the definition of $V(t, t+1) = e^{-\delta}$. □

Definition 8 $d = 1 - e^{-\delta}$ *is called the effective discount rate,* $v = e^{-\delta}$ *the effective discount factor per unit time.*

i, d and v are all expressed in terms of δ, and any one of them determines all the others.

d is sometimes quoted if a bill due at at a future date contains a (simple, so-called commercial) early payment discount proportional to the number of days we pay in advance. In analogy with simple interest this means, that a billed amount C due at time t can be paid off paying $C(1 - td)$.

Returning to the compound interest model, v is very useful in expressions like in the last sections where now

$$v(s) = v^s, \quad V(s,t) = v^{t-s}, \quad A(s,t) = v^{-(t-s)}.$$

Example 17 For a return of £10,000 in 4.5 years' time, how much do you have to invest today at an effective interest rate of 5% p.a.?

$$v = e^{-\delta} = \frac{1}{1+i} = \frac{1}{1.05}, \quad C = v^{4.5} 10,000 = 8028.75.$$

We can also say, that today's value of a payment of £10,000 in 4.5 years' time is £8028.75.

Lecture 4

Fixed-interest securities and Annuities-certain

In this chapter we work out practical examples in the compound interest model, that are of central importance: securities and annuities. We put some emphasis on cases where the payment frequency is not unit time. Often, when choosing to the appropriate time unit, there is no need for a continuous time model and we could switch to the naturally embedded discrete time model. Also, all securities and annuities considered here are assumed to have no risk of default.

4.1 Simple fixed-interest securities

As seen in Example 5, an investment into the simplest type of a fixed-interest security pays interest at rate j at the end of each time unit over an integer term n and repays the invested money at the end of the term. The cash flow representation is

$$c_0 = ((0, -C), (1, Cj), \dots, (n-1, Cj), (n, C + Cj))$$

In practice, a security is a piece of paper (with coupon strips to cash the interest) that can change owner (sometimes under some restrictions). It is therefore useful to split $c_0 = ((0, -C), c)$ into the inflows c and their purchase price at time 0. The intrinsic model for this security is the compound interest model with constant rate $i = j$. The trading price of the security is then $DVal_t(c)$, the discounted value of all post-t flows. At any integer time t after the interest payment this value is C whereas the value increases exponentially between integer times. Note that C is the fair price at time 0 in this model because $Val_0(c_0) = 0$, where we recall $Val = AVal + DVal$.

4.2 Securities above/below/at par

More generally, one can consider securities with coupon payments at rate j different from the not necessarily constant interest rate of the market, or with rates increasing from year to year. In this case, the initial capital C and the repayment sum R do not coincide.

And also interest may be paid on a third, so-called *nominal amount* N. In any given model a security

$$c = ((1, Nj_1), (2, Nj_2), \ldots, (n-1, Nj_{n-1}), (n, R + Nj_n))$$

is then bought at $DVal_0(c)$, which for constant interest rates and constant coupon payments $j_k = j$, $k = 1, \ldots, n$, is

$$DVal_0(c) = jN \sum_{k=1^n} v^k + Rv^n = jNv\frac{1-v^n}{1-v} + Rv^n.$$

If $DVal_0(c) < N$, we say that the security is *below par* or *at a discount*. If $DVal_0(c) > N$, we say that the security is *above par* or *at a premium*. If $DVal_0(c) = N$, we say that the security is *at par*.

If the security is not *redeemed at par*, the redemption price R is stated on the security as a percentage $P = 100R/N$ of the nominal amount N. Interest payments are always calculated from the nominal amount. Redemption at par is the standard.

4.3 $p$thly paid interest

Another generalisation is to increase the frequency of interest payments. This is a very important feature since virtually all British securities have semi-annual coupon payments whereas it is natural to work with annual unit time.

Before valuing securities, we introduce the notion of *nominal* interest rates that is used whenever interest is paid more than once per time unit. There is also an analogue for discount rates.

Example 18 If a bank offers 8% interest per annum convertible quarterly, then it often means that it pays 2% interest per quarter. We check that an initial capital of £10000.00 increases via £10200.00, £10404.00 and £10612.08 to £10824.32 in one year. We have called this an *effective* interest rate of 8.2432%. To distinguish, we call the rate of 8% given in the beginning, the *nominal interest rate convertible quarterly*.

The general concept is as follows.

Definition 9 Given an effective interest rate $i \in \mathbb{R}_+$ and a frequency of $p \in \mathbb{N}$ payments per time unit, we call $i^{(p)}$ such that

$$\left(1 + \frac{i^{(p)}}{p}\right)^p = 1 + i, \quad \text{i.e.} \quad i^{(p)} = p\left((1+i)^{1/p} - 1\right)$$

the *nominal interest rate convertible pthly*.

We can also see $i^{(p)}$ as the total amount of interest payable in equal instalments at the end of each pth subinterval. This formulation should be taken with care, however, since we add up payments made at different times, and their time values are not the same. Also, interest payments should not be mixed up with jumps in value for an investment since our models specify continuous increase in value by interest, and in fact the right on interest is accumulated continuously just with the payment made later (in arrear).

For $p \to \infty$ we obtain

Proposition 8 $\lim_{p\to\infty} i^{(p)} = \delta$, *the force of interest.*

Proof: By definition of $i^{(p)}$, this limit takes the derivative of $t \mapsto (1+i)^t$ at $t=0$. We have done this in Definition 4 to introduce the force of interest. □

4.4 Securities with $p$thly paid interest

A security of term n and (nominal=redemption) value N that pays interest at rate j (nominal) convertible $p$thly is the cash flow

$$c = \left(\left(\frac{1}{p}, \frac{j}{p}N\right), \left(\frac{2}{p}, \frac{j}{p}N\right), \ldots, \left(n - \frac{1}{p}, \frac{j}{p}N\right), \left(n, \frac{j}{p}N + N\right)\right).$$

It can be bought at time 0 for $DVal_0(c)$ which in the compound interest model at constant interest rate i is

$$DVal_0(c) = \left[v^n + \frac{j}{p}\sum_{k=1}^{pn} v^{k/p}\right] N = \left[v^n + \frac{j}{p} v^{1/p} \frac{1-v^n}{1-v^{1/p}}\right] N.$$

4.5 Annuities-certain

As we saw in Example 8, an annuity-certain of term n provides annual payments of some constant amount X:

$$c = ((1, X), (2, X), \ldots, (n-1, X), (n, X)).$$

In the constant rate compound interest model the issue price can be given by

$$DVal_0(c) = X \sum_{k=1}^{n} v^k = Xv\frac{1-v^n}{1-v} = X\frac{1-v^n}{i} =: X\ a_{\overline{n}|}$$

where the last symbol, or more precisely $a_{\overline{n}|i}$ to mention the interest rate, is read 'a angle n (at i)'. This is the first example of the peculiar actuarial notation that has been developped over centuries.

Obviously, this formula also allows to calculate the payment amount X from a given capital to be invested at time 0.

Actuaries also use the following notation for the accumulated value at time n

$$AVal_n(c) = v^{-n} DVal_0(c) = X\frac{(1+i)^n - 1}{i} =: X\ s_{\overline{n}|}.$$

4.6 Perpetuities

Perpetuities are annuities providing payments in perpetuity, i.e.

$$c = ((1, X), (2, X), \ldots, (n, X), \ldots).$$

We have

$$DVal_0(c) = X \sum_{k=1}^{\infty} v^k = X \frac{1}{i} =: X\ a_{\overline{\infty}|}.$$

Note that the value of a perpetuity at integer times remains constant. It can therefore also be seen as a fixed-interest security with an infinite term.

4.7 Annuities and perpetuities payable $p$thly and continuously

Also annuities and perpetuities with a higher frequency of interest payments may be considered. The actuarial symbols are

$$a_{\overline{n}|}^{(p)} = \frac{1}{p} \sum_{k=1}^{pn} v^{k/p} = \frac{v}{p} \frac{1 - v^n}{1 - v^{1/p}} = \frac{1 - v^n}{i^{(p)}}$$

for the present value of payments of $1/p$ p times a year over a term n,

$$s_{\overline{n}|}^{(p)} = v^{-n} a_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{i^{(p)}}$$

for the accumulated value at time n and

$$a_{\overline{\infty}|}^{(p)} = \frac{1}{p} \sum_{k=1}^{\infty} v^{k/p} = \frac{v}{p} \frac{1}{1 - v^{1/p}} = \frac{1}{i^{(p)}}$$

for the corresponding perpetuity.

If we pass to the limit of p to infinity, we obtain continuously payable annuities and perpetuities, and formulas

$$\begin{aligned}
\bar{a}_{\overline{n}|} &= \int_0^n v^t dt = \frac{v^n - 1}{log(v)} = \frac{1 - v^n}{\delta} \\
\bar{s}_{\overline{n}|} &= v^{-n} \bar{a}_{\overline{n}|} = \frac{1 - v^n}{v^n \delta} \\
\bar{a}_{\overline{\infty}|} &= \frac{1}{\delta}.
\end{aligned}$$

Note the similarity of all these expressions for ordinary, $p$thly payable and continuously payable annuities, and one could add more. They only differ in the appropriate interest rate i, $i^{(p)}$ or δ. This can be understood by an equivalence principle for interest payments: the payment of i at time 1 is equivalent (has the same value) to p equally spread payments of $i^{(p)}/p$ or an equally spread continuous payment at rate δ.

Lecture 5

Mortgages and loans

As we indicated in the Introduction, interest-only and repayment loans are the formal inverse cash flows of securities and annuities. Therefore, most of the last lecture can be reinterpreted for loans. We shall here only translate the most essential formulae and then pass to specific questions and features arising in loans and mortgages, e.g. calculations of outstanding capital, proportions of interest/repayment, discount periods and APR.

5.1 Loan repayment schemes

A *repayment scheme* for a loan of amount L at force of interest $\delta(\cdot)$ is a cash flow

$$c = ((t_1, X_1), (t_2, X_2), \ldots, (t_n, X_n))$$

such that

$$L = DVal_0(c) = \sum_{k=1}^{n} v(t_n) X_n. \tag{1}$$

Condition (1) ensures that, in the model given by $\delta(\cdot)$, the loan is repaid after the nth payment since it means that the cash flow $((0, -L), c)$ has zero value at time 0, and then by Corollary 2 at all times.

Example 19 A bank lends you £1000 at an effective interest rate of 8% p.a. initially, but due to rise to 9% after the first year. You repay £400 both after the first and half way through the second year and wish to repay the rest after the second year. The first two payments are worth $400(1.08)^{-1} = 370.37$ and $400(1.09)^{-1/2}(1.08)^{-1} = 354.75$ at time 0, hence the final payment $274.88 = 323.59(1.09)^{-1}(1.08)^{-1}$, £323.59 after two years.

Often, the times t_k are regularly spaced and passing to the appropriate unit time, we can assume $t_k = k$. Often, the interest rate is constant i say and the payments are level payments X. This is an inverse ordinary annuity and

$$L = X \; a_{\overline{n|}}$$

allows to calculate X from L, n and i.

Let us return to the general case. In our example, we compared values at time 0 to calculate the outstanding debt, an important quantity. In general we have the following so-called *retrospective formula*.

Proposition 9 *Given a loan* $(L, \delta(\cdot))$ *and repayments*

$$c_m = ((t_1, X_1), (t_2, X_2), \ldots, (t_m, X_m))$$

up to time t*, the outstanding debt is*

$$AVal_t((0, L)) - AVal_t(c_m) = A(0, t)L - \sum_{k=1}^{m} A(t_k, t)X_k =: L_t.$$

Proof: The equation says that the time-t value of the loan minus the time-t values of all previous payments is the outstanding loan. Hence, $((0, -L), c_m, (t, L_t))$ is a zero-value cash flow as required. □

Alternatively, for a given repayment scheme (satisfying (1)), one can also use the following *prospective formula*.

Proposition 10 *Given a loan* $(L, \delta(\cdot))$ *and a repayment scheme* c*, the outstanding debt at time* $t \in [t_j, t_{j+1})$ *is*

$$L_t = DVal_t(c) = \sum_{k=j+1}^{n} V(t, t_k)X_k. \tag{2}$$

Proof: By assumption, $((0, -L), c)$ is a zero-value cash flow. If we call the right hand side of (2) R, then $((t, -R), c_{(t,\infty)})$ is a zero-value cash flow, and so is their difference. Hence R repays the loan at time t, $L_t = R$. □

It is important to have both a mathematical understanding of the model that allows to do explicit calculations, and a practical understanding to argue 'by general reasoning'. The two preceding proofs are more of the latter style, although still rigorous. A more mathematical proof can be given by induction over the number of payments.

Corollary 3 *The* j*th payment of a loan repayment schedule* c *consists of* $R_j = L_{t_{j-1}} - L_{t_j}$ *capital repayment and* $I_j = X_j - R_j = L_{t_{j-1}}(A(t_{j-1}, t_j) - 1)$ *interest payment.*

5.2 Equivalent cash flows and equivalent models

In practice, the embedded discrete time model is more important than our more general continuous model. Often, cash flows can be simplified onto a discrete lattice. One method to achieve discrete time models from continuous time models is to simplify cash flows by moving them onto a lattice, i.e. replacing them by suitable cash flows that only contain transaction on a time grid.

Definition 10 *Given a model* $\delta(\cdot)$*, two cash flows* c_1 *and* c_2 *are called* equivalent in $\delta(\cdot)$ *if* $Val_t(c_1) = Val_t(c_2)$ *for one (all)* $t \in \mathbb{R}_+$.

In this sense, all repayment schemes of a loan in a given model are equivalent. Also, in a constant interest rate model, $p$thly interest payments at nominal rate $i^{(p)}$ are equivalent for all $p \in \mathbb{N}$, and they are equivalent to continuous interest payments at rate δ. Note that the equivalence of two cash flows depends on the model. In fact, two cash flows that are equivalent in all models, are the same.

Proposition 11 $a_{\overline{n}|}^{(p)} = \frac{i}{i^{(p)}} a_{\overline{n}|}$

Proof: By the definition of $i^{(p)}$, $p$thly level payments of $i^{(p)}/p$ are equivalent to payments i per time unit, that is $p$thly level payments of $1/p$ are equivalent to payments $i/i^{(p)}$ per unit time. Extended over n time units, they define $a_{\overline{n}|}^{(p)}$ and $(i/i^{(p)})a_{\overline{n}|}$ respectively. □

This gives an alternative approach to $p$thly payable annuities.

Definition 11 *Given a cash flow c, two models $\delta_1(\cdot)$ and $\delta_2(\cdot)$ are called* equivalent *for c if $\delta_1 - Val_t(c) = \delta_2 - Val_t(c)$ for all $t \in supp(c)$, where $supp(c) := \{0, t_1, t_2, \ldots, t_n\}$ for a discrete cash flow $c = ((t_1, C_1), (t_2, C_2), \ldots, (t_n, C_n))$.*

Note that the equivalence of models depends on the cash flow. Two models that are equivalent for all cash flows coincide.

Proposition 12 *Given a discrete cash flow c and a model $\delta_1(\cdot)$, a model $\delta_2(\cdot)$ is equivalent for c if and only if for all $j = 0, \ldots, n-1$*

$$\int_{t_j}^{t_{j+1}} \delta_1(s)ds = \int_{t_j}^{t_{j+1}} \delta_2(s)ds.$$

In particular, there is always a piecewise constant model δ_3 equivalent to δ_1 for c.

Proof: For a discrete cash flow the value of c at time t_j is given by

$$Val_{t_j}(c) = \frac{1}{v(t_j)} \sum_{k=1}^{n} v(t_k)c_k.$$

Provided, this value coincides for $j = 0$, equality for all other j enforces $v_1(t_j) = v_2(t_j)$ for all $j = 1, \ldots, n$. Vice versa, if $v_1(t_j) = v_2(t_j)$ for all $j = 1, \ldots, n$, then values coincide for all $j = 0, \ldots, n$. Now by definition

$$v_1(t_j) = \exp\left\{ -\int_0^{t_j} \delta_1(s)ds \right\} = v_2(t_j)$$

for all $j = 1, \ldots, n$ if and only if the integrals of δ_1 and δ_2 from 0 to t_j coincide for all $j = 1, \ldots, n$ if and only if the integrals from t_{j-1} to t_j coincide for all $j = 1, \ldots, n$.

For the second assertion just define for $j = 1, \ldots, n$

$$\delta_3(s) = \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} \delta_1(t)dt, \qquad t_j \le t < t_{j+1}.$$

□

Proposition 13 *Piecewise constant models $\delta(\cdot)$ can be represented by $i_j = e^{\delta_j} - 1$ via*

$$v(t) = (1+i_0)^{-t_1}(1+i_1)^{-(t_2-t_1)} \dots (1+i_j)^{-(t-t_j)}, \qquad t_j \le t < t_{j+1}$$

if $\delta(t) = \delta_j$ for all $t_j < t < t_{j+1}$, $j \in \mathbb{N}$.

Proof: By definition of $v(t)$

$$v(t) = \exp\left\{-\int_0^t \delta(s)ds\right\} = \exp\left\{-\sum_{k=0}^{j-1}(t_{k+1}-t_k)\delta_k - (t-t_j)\delta_j\right\}$$

which transforms to what we need via $1 + i_j = e^{\delta_j}$. □

Sometimes, evaluating a cash flow c in a model $\delta(\cdot)$ can be conveniently carried out passing first to an equivalent cash flow and then to an equivalent model.

5.3 Fixed, discount, tracker and capped mortgages

In practice, the interest rate of a mortgage is rarely fixed for the whole term and the lender has some freedom to change their Standard Variable Rate (SVR). Usually changes are made in accordance with changes of the UK base rate fixed by the Bank of England. However, to attract customers, an initial period has often some special features.

Example 20 (Fixed period) For an initial 2-10 years, the interest rate is fixed, usually below the current SVR, the shorter the period, the lower the rate.

Example 21 (Capped period) For an initial 2-5 years, the interest rate can fall parallel to the base rate or the SVR, but cannot rise above the initial level.

Example 22 (Discount period) For an initial 2-5 years, a certain discount on the SVR is given. This discount may change according to a prescribed schedule.

Example 23 (Tracking period) For an initial or the whole period, the interest rate moves parallel to the UK or another base rate rather than following the lender's SVR.

Whichever special features there may be, the monthly payments are always calculated as if the current rate was valid for the whole term. Therefore, even if the rate is known to change after an initial period, no level payments are calculated. The effect is that, e.g. a discount period leads to lower initial payments. With every change in interest rate (whether known in advance or reacting on changes in the base rate) leads to changes in the monthly payments. In some cases, there may be the option to keep the original amount and extend the term.

Initial advantages in interest rates are usually combined with early redemption penalties that may or may not extend beyond the initial period. A typical penalty is 6 months of interest on the amount redeemed early.

Lecture 6

An introduction to yields

Given a cash flow representing an investment, its yield is the constant interest rate that makes the cash flow a fair deal. Yields allow to assess and compare the performance of possibly quite different investment opportunities as well as mortgages and loans.

6.1 Flat rates and APR

To compare different mortgages with different features, two common methods should be mentioned, a bad one and a better one. The bad method is the so-called *flat rate* which is the total interest per year of the loan per unit of initial loan, i.e.

$$F = \frac{\sum_{j=1}^{n} I_j}{t_n L} = \frac{\sum_{j=1}^{n} X_j - L}{t_n L}$$

where usually $X_j = X$, $t_n = n$. This method is bad because it does not take into account that as time evolves, interest is paid only on the outstanding loan. One consequence is that loans with different term but same interest rates can have very different flat rates.

The better method is to give the Annual Percentage Rate of Charge (APR). In case of a fixed (effective) interest rate i, this is just i. If the interest rate varies, APR is the constant rate under which the schedule exactly repays the loan, rounded to the next lower 0.1%. We call this the yield and subject of the next section. Note that the constant interest rate model is not equivalent for the schedule, in general, since we do not and cannot expect that values coincide at all payment dates.

Example 24 *Given a mortgage of amount L over a term of 25 years with a discount period of 5 years at 3%, after which the SVR of currently 6% is payable. Although this is not common practice, let us assume that level payments $c = ((1, X), \ldots, (25, X))$ are made over the whole term. Then*

$$L = DVal_0(c) = Xa_{\overline{5}|3\%} + X(1.03)^{-5}a_{\overline{20}|6\%}$$

determines X, and the APR is essentially i such that

$$L = i\text{-}DVal_0(c) = Xa_{\overline{25}|i}.$$

This cannot be solved algebraically, but numerically we obtain $i \approx 4.737\%$, hence $APR =$ 4.7%

6.2 The yield of a cash flow

Suppose, we are offered a deal that involves transactions according to a cash flow c. So far we have learnt how to calculate the present value in a given model. This tells whether the deal is profitable ($Val_0(c) > 0$) or not. However, in practice there does not exist a true known model. Whoever offers us the deal, may have a model δ_1 in mind that is different from our model δ_0 (because of different expectations of interest rate evolution, to mention the simplest influence). One can introduce a concept of utility to explain the coexistence of different models (e.g. you may feel that the first £10,000 are worth more than the second £10,000), but we shall not follow this route here. The essential question remains: do we believe in one model δ_0? Maybe it is prudent to compare different models.

Given a cash flow c, it is rather hopeless to look at the family of values for all time-dependent forces of interest $\delta(\cdot)$, but focussing on constant interest rate models, $i \in \mathbb{R}_+$, gives us a nice family of values that we denote $a_i = i\text{-}Val_0(c)$. We can draw this as a function of i to show us under what interest rate assumptions the deal c is profitable.

Before we pass to any definite statements, one remark on the class of cash flows that we look at. In that follows, it does not make much difference whether c is discrete, continuous or mixed, whether c is finite (i.e. has a last in- or outflow) or infinite (like e.g. perpetuities). Only, to reasonably include the infinite case, we assume in the sequel that the cash flows are *finite values cash flows.*

Proposition 14 *Given a cash flow c the function $i \mapsto a_i = i\text{-}Val_0(c)$ is continuous on $(-1, \infty)$.*

Proof: In the discrete case $c = ((t_1, C_1), \ldots, (t_n, C_n))$

$$a_i = i\text{-}Val_0(c) = \sum_{k=1}^{n} C_k v(t_k) = \sum_{k=1}^{n} C_k (1+i)^{-t_k}.$$

and this is clearly continuous in i for all $i > -1$. For a continuous-time cash flow $c(s)$, $0 \leq s \leq t$ we use the uniform continuity of $i \mapsto (1+i)^{-s}$ on compact intervals $s \in [0, t]$ for continuity to be maintained after integration

$$a_i = i\text{-}Val_0(c) = \int_0^t c(s)v(s)ds = \int_0^t c(s)(1+i)^{-s}ds.$$

$\square$

Corollary 4 *For any cash flow c and time t the function $i \mapsto i\text{-}Val_t(c)$ is continuous on $(-1, \infty)$.*

Often the situation is such that a deal is profitable if the interest rate is below a certain level, but not above, or vice versa. By the intermediate value theorem, this threshold is a zero of $i \mapsto a_i$, and we define

Definition 12 *Given a cash flow c, if $i \mapsto a_i = i\text{-}Val_0(c)$ has a unique root on $(-1, \infty)$, we define the yield $y(c)$ to be this root. If $i \mapsto a_i$ does not have a root in $(-1, \infty)$ or has more than one root, we do not define the yield of this cash flow.*

So far, we implicitly or explicitly assumed that $i \geq 0$, both to ease intuition and to streamline some minor technical issues. Particularly in this setting of yields, we should allow for the case that money invested in certain projects can result in a loss, which amounts to a negative interest rate. All formulas remain the same, provided $i > -1$. In fact, $i = -1$ creates singularities in the discount factors because it corresponds to a complete loss of money, so that the original value is no longer a fraction of the later value (zero).

The concept is useful only because of the continuity of $i \mapsto a_i$. If this function was not continuous, profitability could change without root.

The yield can be interpreted as the fixed interest rate under which the deal c is fair. We see in the next section that the yield exists for the majority of practical situations. The yield is also called the *internal rate of return* or the *money-weighted rate of return.* For the assessment of funds there are other weighted rates of return that we discuss in a later lecture.

Example 25 Suppose that for an initial investment of £1000 you obtain a payment of £400 after one year and 770 after two years. What is the yield of this deal? By definition, we are looking for zeros $i \in (-1, \infty)$ of

$$\begin{aligned} a_i = -1000 + 400(1+i)^{-1} + 800(1+i)^{-2} = 0 \\ \iff \qquad 1000(i+1)^2 - 400(i+1) - 770 = 0 \end{aligned}$$

The solutions to this quadratic equation are $i_1 = -1.7$ and $i_2 = 0.1$. Since only the second zero lies in $(-1, \infty)$, the yield is $y(c) = 0.1$, i.e. 10%.

6.3 General results ensuring the existence of the yield

Proposition 15 *If c has in- and outflows and all inflows of c precede all outflows of c (or vice versa), then the yield $y(c)$ exists.*

Proof: By assumption, there is (at least one) $T \in \mathbb{R}_+$ such that all inflows are in $[0, T)$ and all outflows in (T, ∞). Then

$$p_i = i\text{-}AVal_T(c)$$

is positive strictly increasing in i with $p_{-1} = 0$ and $p_\infty = \infty$ (by assumption there are inflows) and

$$n_i = i\text{-}DVal_t(c)$$

is negative strictly increasing with $n_{-1} = -\infty$ (by assumption there are outflows) and $n_\infty = 0$. Therefore

$$b_i = p_i + n_i = i\text{-}Val_T(c)$$

is strictly increasing from $-\infty$ to ∞, continuous by Corollary 4, denote the unique root by i_0. Then i_0 is also the unique root of $a_i = i\text{-}Val_0(c) = (1+i)^{-T}\,[i\text{-}Val_T(c)]$ by Corollary 2. $\square$

This result applies to all typical capital investment and borrowing situations.

Corollary 5 *In the situation of the Proposition 15 with inflows preceding outflows* $y(c) > 0$ *if and only if* $0Val_0(c) < 0$. *If outflows precede inflows* $y(c) > 0$ *if and only if* $0\text{-}Val_0(c) > 0$.

Proof: In the first setting assume $y(c) > 0$, then by the monotonicity of $i \mapsto b_i$ we have $b_0 < 0$ and therefore $0\text{-}Val_0(c) = a_0 = (1+i)^{-T} b_0 < 0$.

If conversely $a_0 = 0\text{-}Val_0(c) < 0$ then also $b_0 < 0$ and by the intermediate value theorem with $b_0 < 0$ and $b_\infty = \infty$ the root lies between 0 and ∞.

The second setting is analogous with the obvious changes in signs. □

Proposition 16 *If* $t \mapsto 0\text{-}AVal_t(c)$ *(corresponds to zero-interest!) changes sign precisely once, then there is a unique positive root to* $i \mapsto a_i$.

Proof: Let T be the time of the sign change (any choice if it happens around an interval of zeros). W.l.o.g. this sign change is from plus to minus. Then $0\text{-}Val_0(c) < 0$, hence $b_0 = 0\text{-}Val_T(c) < 0$. Now see what happens for

$$b_i = i\text{-}Val_T(c)$$

when i tends to infinity. We split again into past and future:

$$p_i = i\text{-}AVal_{T-}(c)$$

is positive and increases strictly to infinity as i tends to infinity (more and more interest on a changing but positive balance).

$$n_i = i\text{-}DVal_{T-}(c)$$

is negative and increases strictly to zero (more and more discount on a changing but negative balance). Therefore b_i strictly increases to infinity as i tends to infinity. Since $b_0 < 0$, there must be a zero of $i \mapsto b_i$ in $(0, \infty)$. □

It is often useful to refer to this unique positive root as the yield even if the existence of negative roots cannot be excluded. A practical example is, if a mortgage balance is increased during the term (e.g. to finance major refurbishing). The proposition does not apply in this general situation, but often it does. As a slight modification one might replace the zero interest lower bound by the minimal interest rate payable throughout the term. A similar result holds and establishes a yield for this situation:

Corollary 6 *If (under interest rate* i_0*)* $t \mapsto i_0\text{-}AVal_t(c)$ *changes sign precisely once, then* $i \mapsto a_i$ *has a unique root* $i \geq i_0$.

Proof: To adapt the proof of Proposition 16, replace $0\text{-}Val.(c)$ by $i_0\text{-}Val.(c)$ throughout. □

In the next lecture we shall see some more applications of yield calculations.

Lecture 7

Project appraisal

Last lecture we introduced the yield of cash flows and indicated how it can be used to assess investment projects. We make this more precise here and discuss methods that are used in practice for specific types of projects. These include calculating payback periods for business projects and weighted rates of return for investment funds.

7.1 A remark on numerically calculating the yield

Let c be a cash flow. Suppose Proposition 15 guarantees the existence of the yield $y(c)$. Remember, this means that $f(i) = i\text{-}Val_0(c)$ is continuous, strictly monotone and takes values of different signs at the boundaries of $(-1, \infty)$.

Interval splitting allows to trace the root of f: $(l_0, r_0) = (-1, \infty)$, make successive guesses $i_n \in (l_n, r_n)$, calculate $f(i_n)$ and define

$$(l_{n+1}, r_{n+1}) := (i_n, r_n) \quad \text{or} \quad (l_{n+1}, r_{n+1}) = (l_n, i_n)$$

such that the values at the boundaries $f(l_{n+1})$ and $f(r_{n+1})$ are still of different signs.

The challenge is to make good guesses. Bisection

$$i_n = (l_n + r_n)/2$$

(once $r_n < \infty$) is the ad hoc way, *linear interpolation*

$$i_n = \frac{r_n f(r_n) - l_n f(l_n)}{f(r_n) - f(l_n)}$$

an efficient improvement. There are more efficient variations of this method using some kind of convexity property of f, but we don't go beyond these indications here.

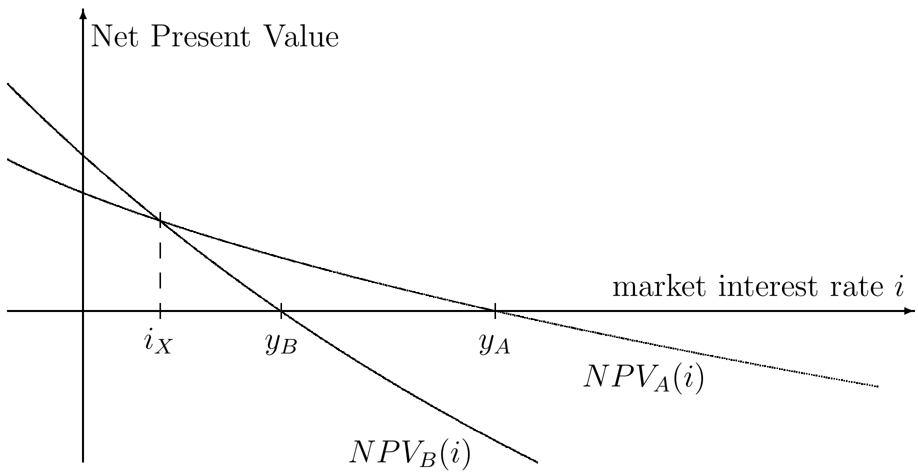
7.2 Comparison of investment projects

First, since the notation $i\text{-}Val_0(c)$ is getting a bit heavy, we introduce new (actually more standard) notation. For a cash flow c, or cash flows c_A and c_B representing two investment projects

$$NPV(i) := i\text{-}Val_0(c), \quad NPV_A(i) := i\text{-}Val_0(c_A) \quad \text{and} \quad NPV_B(i) := i\text{-}Val_0(c_B).$$

These quantities are called the Net Present Values of the underlying cash flows at interest rate i. 'Net' may seem a bit odd in our context, but refers to the fact that both in- and outflows have been incorporated. We have done this from the very beginning.

Now to compare projects A and B that satisfy the conditions of Proposition 15 say, one can calculate their yields y_A and y_B. Suppose you have first the outflows and then the inflows, then each project is profitable if its yield exceeds the market interest rate. In particular if the market interest rate is in (y_A, y_B), say, project B is profitable, project A is not. But this does not mean that project B is more profitable than project A (i.e. $NPV_B(i) > NPV_A(i)$ for all lower interest rates as the following figure shows.



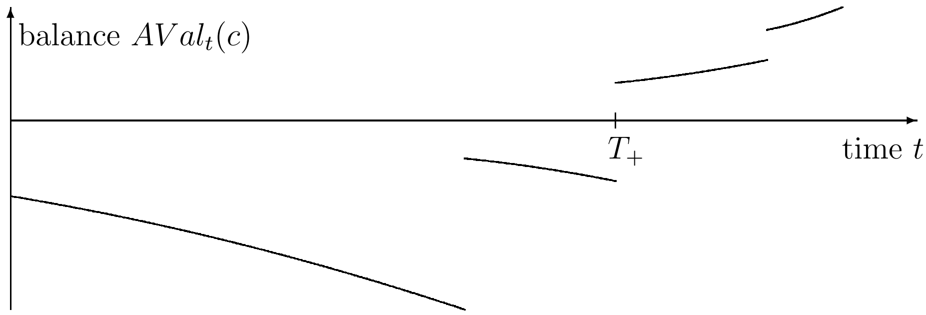
i_X is called a *cross-over rate* and can be calculated as the yield of $c_A - c_B$ if it is unique. For interest rates below i_X, project B is more profitable than project A, although its yield is smaller. A decision for one or the other project (or against both) now clearly depends on the expectations on interest rate changes.

7.3 Investment projects and payback periods

For a profitable investment project in a given interest rate model $\delta(\cdot)$ we can study the evolution of the accumulated value. With outflows preceding inflows, this accumulated value will first be negative, but tend to the positive terminal value (which is positive since the project is profitable). Of interest is the time when it becomes first positive.

Definition 13 Given a model $\delta(\cdot)$ and a profitable cash flow c with outflows preceding inflows. We define the *discounted payback period*

$$T_+ = \inf\{t \geq 0 : AVal_t(c) \geq 0\}.$$



If rather than investing existing capital, you finance the investment project by taking out loans using any inflows for repayment, T_+ is the time when your account balance changes from negative to positive. Therefore, you have repaid your debt and all remaining inflows are simply profit.

Obviously, before time T_+ you pay interest, after time T_+ you receive interest. If we now specify a model for an account, say, to keep it simple, that has different forces of interest δ_- when the balance is negative and δ_+ when the balance is positive, then the definition of T_+ only depends on δ_- and the profit can be calculated from δ_--$AVal_{T_+}(c)$ and δ_+-$DVal_{T_+}(c)$.

Lecture 8

Funds and weighted rates of return

Funds are pools of money into which people pay for different reasons and different benefits. They may be simply investment opportunities. In this case they usually clearly state what proportions they invest into securities and/or equities of certain countries or branches of the economy with some freedom remaining. Alternatively they may consist of the reserves of pension schemes or contain foundation capital. In any of the cases there is a fund manager who adapts the investments to current market situations and releases any money withdrawn from the fund.

An important issue is to assess the performance of a fund, for instance to check that the fund manager does a good job. Also, in the case of an investment fund, investors need to know the exact value of their capital invested in the fund.

Definition 14 Let F_0 be the initial amount of a fund, c the cash flow describing its in- and outflows between times 0 and T, F_T the amount at time T. The *money weighted rate of return* of the fund between times 0 and T is defined to be the yield $y(c_{0T})$ of the cash flow

$$((0, F_0), c, (T, -F_T)).$$

But this is not the yield for the investor:

Example 26 Suppose there is a single investor who invests £100 into a fund at time 0 and receives £150 at time 4. His yield is easily calculated from the equation of value

$$100 - 150(1 + y_1)^{-4} = 0 \qquad \Rightarrow \qquad y_1 \approx 10.668\%.$$

Suppose, in between a second investor invested £100 at time 1 when the value of the investment of the first investor so happened not to have changed. At time 2 however, the fund had lost 25%, and the second investor decides to withdraw his share at time 3 when again, the value of the investment did not change again. He obtained therefore £75. As noted above, the value of the first investor's investment rises to £150 by time 4 which is an increase of 100% since time 3. Now, we solve the yield equation

$$100 + 100(1 + i)^{-1} - 75(1 + i)^{-3} - 150(1 + i)^{-4} = 0$$

to obtain (numerically) a yield of $y_{1+2} \approx 3.789\%$.

This value is much lower since it takes into account the loss that the second investor experienced.

By the way, to assess the fund manager, the money weighted rate of return is not quite fair either since he has no influence on the in- and outflows. If in- and outflows form a discrete cash flow, we can calculate the yield between any two successive flows. In these periods, the evolution reflects the skills of the fund manager.

Definition 15 Given a fund of amount F_t at time $t \in [0, T]$ with in- and outflows according to $c = ((t_1, C_1), \ldots, (t_n, C_n))$, we define the *time weighted rate of return* to be the value $i \in (-1, \infty)$ such that

$$(1+i)^T = \frac{F_{t_1-}}{F_{0+}} \frac{F_{t_2-}}{F_{t_1+}} \cdots \frac{F_{t_n-}}{F_{t_{n-1}+}} \frac{F_{T-}}{F_{t_n+}}.$$

Proposition 17 *The time weighted rate of return is the yield between* 0 *and* T *for an investor who invested at time* 0*.*

Proof: The statement is true if there is no flow. We now interpret the factors $F_{t_j-}/F_{t_{j-1}+}$ as accumulation factors. An initial capital of C in the fund accumulates to

$$A_1 = C\frac{F_{t_1-}}{F_{0+}}$$

by time t_1. The fact that other investors pay into or withdraw from the fund, does not change our investor's accumulated value A_1. We can therefore just repeat the argument and multiply by the next ratio to calculate the accumulated value A_2 at time t_2, and so on, until t_n and eventually T where we get an accumulated value of

$$S = C\frac{F_{t_1-}}{F_{0+}} \frac{F_{t_2-}}{F_{t_1+}} \cdots \frac{F_{t_n-}}{F_{t_{n-1}+}} \frac{F_{T-}}{F_{t_n+}}.$$

Now the yield of this investment $c_T = ((0, -C), (T, S))$ is defined as i such that

$$0 = NPV(i) = -C + (1+i)^{-T} = -C + (1+i)^{-T} C\frac{F_{t_1-}}{F_{0+}} \frac{F_{t_2-}}{F_{t_1+}} \cdots \frac{F_{t_n-}}{F_{t_{n-1}+}} \frac{F_{T-}}{F_{t_n+}}$$

and this is the equation for the time weighted rate of return. □

The time weighted rate of return has its name since it is an average of yields between flows that takes into account the lengths of these time periods. The money weighted rate of return gives more weight to times when the fund amount is big. So far, we have only pointed out where the money weighted rate of return is *not* suitable. We should stress that it does reflect the (money weighted, as it should be) average yield of investors.

Lecture 9

Inflation

9.1 Inflation indices

Inflation means goods are getting "more expensive". An inflation index simply records these prices for a particular good or a basket of goods. The most commonly used index is the Retail Price Index (RPI). Its basket contains virtually everything, from different kinds of bread over salaries and houses to electricity and gas, whatever an average Englishman is likely to spend, weighted in a way to express statistical relevance.

Year	1996	1997	1998	1999	2000	2001	2002
Retail Price Index in January	150.2	154.4	159.5	163.4	166.6	171.1	173.3
Annual Inflation rate	2.8%	3.3%	2.4%	2.0%	2.7%	1.3%	

where the inflation rates are calculated e.g. as $e_{2001} = 1 + 1.3\% = 173.3/171.3$. We will see later how these inflation rates can be incorporated into our interest models. Theoretically, e_k is the interest rate you earn by buying the basket at time k and sell it at time $k + 1$. (In practice, many goods in the basket don't allow this.)

Although we will probably exclusively work with the RPI, it should be noted that there are other important indices that are worth mentioning. Also, prices for any specific good are likely to behave in a completely different way from the RPI. When buying a house, you may wish to consult the House Price Index (current inflation rates are around 20% and buyers fear that house prices start sinking - deflation, negative inflation rates). As a pensioner you have different needs and there is an inflation index that takes this into account (no salaries, no mortgage rates, more weight on medical expenses etc.).

We use the RPI in the context of investments where the primary aim is to accumulate money, and it is important to take account of a general decrease in value due to inflation. Here the RPI is a suitable index, since it reflects all expenses of the whole society.

Example 27 In January 2000 an investor put £1000 in a savings account at a (net) effective rate of 2% interest. His balance in January 2001 was £1020. The RPI tells us that the basket that costed $\pounds_{2000}166.60$ in 2000 costed $\pounds_{2001}171.10$ in 2001. So the $\pounds_{2001}1020$ are effectively only

$$\pounds_{2000}1020 * 166.60/171.10 = \pounds_{2000}1020\frac{1}{1+e_{2000}} = \pounds_{2000}993.17$$

and we say that the real effective interest rate was -0.7%.

Like in physics, it helps to state explicitly the units. Therefore we introduced $£_{2000}$ as the appropriate unit for value in 2000.

For practical reasons, inflation indices only give monthly index values $Q(m/12)$, and if ever this time scale is not fine enough, one uses interpolation. Clearly, a constant rate of inflation in the vague sense above corresponds to an exponentially growing index function Q. Therefore, it is natural, even for non-constant rates, to use exponential interpolation, i.e. linear interpolation on $ln(Q(\cdot))$.

Only the ratio of an index function at two different times enters inflation calculations. Index functions Q can therefore be standardized so as to be 100 or 1 on a specified day.

You may, of course ask, why bother with inflation? We are all used to inflation and we take it intuitively into account when we want to achieve certain yields. One simple answer is that variability in annual yields may be explained by inflation, to some extent. More importantly, there are concepts that require attention to inflation. In funds that contain foundation capital, for instance, the fund manager must aim for a constant (or increasing) real foundation capital, otherwise, the foundation has to reduce its scope. Also, as we shall analyse in more detail in our next lecture, tax authorities claim Capital Gains Tax. Now, if you invested £10,000 in shares in 2002, and sell these shares for £20,000 this year, you are liable to pay 30% tax, £3,000 to the fiscus. If your investment of £10,000 was made in 1980, you have every right to say, this is unfair, since the RPI about doubled as well during this period. This is indeed taken into account.

9.2 Modelling inflation

So far, we have introduced inflation indices and showed how they can be used to take account of varying purchasing power of money when calculating future values of investments. We shall here present a more systematic study of inflation and describe a general model.

From a theoretical perspective, we may consider any positive function Q.

As an inflation index $Q(t)$ merely represents the accumulated value of a certain basket of goods, it has the same structure as $A(0, t)$ in any interest model. It is natural to impose similar regularity conditions and define a force of inflation from $Q(t)$ in the same way as the notion of the force of interest is related to $A(0, t)$.

Definition 16 Let an inflation index function Q be of the form

$$Q(t) = \exp\left\{\int_0^t \gamma(s)ds\right\}$$

for a locally Riemann integrable function $\gamma : [0, \infty) \to \mathbb{R}$. Then we call γ the force of inflation.

In practice, we need both inflation indices and investment accumulation factors since the two are *not* the same: accumulation factors should increase faster than inflation indices, at least in the long run. Otherwise, loans would repay themselves automatically, not in absolute, but in real terms.

Definition 17 Given an interest rate model $\delta(\cdot)$ and an inflation model (force of inflation) $\gamma(\cdot)$, we call $\delta - \gamma$ the (time-dependent) real force of interest

In the sequel we will call a model given by a force of interest δ and a force of inflation γ a (δ, γ) model. For the valuation of cash flows in this model, the following are useful notions:

Definition 18 For a cash flow $c = ((t_1, C_1), \ldots, (t_n, C_n))$ (on the paper), we define the cash flow *net of inflation*

$$c_Q = \left(\left(t_1, \frac{Q(0)}{Q(t_1)} C_1 \right), \ldots, \left(t_n, \frac{Q(0)}{Q(t_n)} C_n \right) \right)$$

that contains everything in time-0 money units (as it feels).

We define the *real yield* $y_R(c) = y(c_Q)$.

Note that we can eliminate the common $Q(0)$ factors in the real yield equation

$$NPV_Q(y) = 0.$$

Indeed, in our model $Q(0) = 1$, but in practice we may wish to choose time 0 different from the normalisation date for Q.

Definition 19 For a cash flow $c = ((t_1, C_1), \ldots, (t_n, C_n))$ (as we want to feel it), we define the *inflation-adjusted cash flow*

$$c^Q = \left(\left(t_1, \frac{Q(t_1)}{Q(0)} C_1 \right), \ldots, \left(t_n, \frac{Q(t_n)}{Q(0)} C_n \right) \right)$$

(on the paper).

Note that $(c_Q)^Q = c$ and $(c^Q)_Q = c$.

Proposition 18 *The following formulas justify our inflation adjustments:*

$$\begin{aligned} NPV(\delta(\cdot)) &= NPV_Q(\delta(\cdot) - \gamma(\cdot)) \\ NPV^Q(\delta(\cdot)) &= NPV(\delta(\cdot) - \gamma(\cdot)). \end{aligned}$$

Proof: For the first, write

$$\begin{aligned} NPV(\delta(\cdot)) &= \sum_{k=1}^{n} \exp\left\{-\int_0^{t_k} \delta(s)ds\right\} C_k \\ &= \sum_{k=1}^{n} \exp\left\{-\int_0^{t_k} (\delta(s)-\gamma(s))ds\right\} \frac{Q(0)}{Q(t_k)} C_k \\ &= NPV(\delta(\cdot)-\gamma(\cdot)). \end{aligned}$$

The second is similar. □

The (δ, γ) model is not equivalent to the $(\delta - \gamma)$ model since a (time-0) valuation of any cash flow has to be made in δ, only the evolution of real values follows $(\delta - \gamma)$. The (δ, γ) model is equivalent to a $(\delta - \gamma)$ model with first all in- and outflows re-expressed in time 0 money units.

Definition 20 *For any force of interest γ and any $t \in [1, \infty)$, we define the inflation rate between $t-1$ and t as*

$$e_t = \exp\left\{\int_{t-1}^{t} \gamma(s)ds\right\} - 1 = \frac{Q(t)}{Q(t-1)} - 1.$$

Often, instead of $Q(m/12)$, inflation statistics show annual rates $e_{m/12}$. We calculated these from the Q values of the RPI in the last lecture.

9.3 Constant inflation rate

A constant inflation rate e means (average) prices of goods increased by a rate e within any one year, i.e. $Q(t+1) = (1+e)Q(t)$ for all t, hence $Q(t) = (1+e)^t$ after normalisation. Clearly e is constant in particular if γ is constant and then $e = \exp\{\gamma\} - 1$.

Now if γ and e, and δ and i, then so is the real force of interest $\delta - \gamma$, and we can associate a real rate of interest:

Definition 21 Given a constant (i, e) model, $j = (1+i)/(1+e) - 1 = (i-e)/(1+e)$ is called the *real interest rate.*

Under a constant inflation rate e, real yields and yields satisfy the same relationship as real interest rates and interest rates:

Proposition 19 *Let c be a cash flow with yield $y(c)$ and a constant inflation rate e. Then the real yield of c exists and is given by*

$$y_e(c) = \frac{y(c) - e}{1+e}.$$

Proof: By definition, the real yield of c is defined as the unique solution to $NPV_Q(y_e) = 0$. By Proposition 18, this is the same as $NPV(y(c)) = 0$ where $y(c)$ and $y_e(c)$ are related as claimed, cf. the above definition. □

Lecture 10

Taxation

In practice, taxation causes complications. From our model's point of view this is not crucial. The changes are mainly to interest rates and amendments at the end of calculations. Inflation is more fundamental. We have to extend our model to include a proper representation. In this lecture we only discuss inflation indices and their use. The more formal model extension is subject of the next lecture.

10.1 Fixed interest securities and running yields

We discussed in Lecture 4 the pricing of fixed interest securities given a model. Clearly these formulas can be used as yield equations to determine the yield given the price.

In the context of securities, the use precise terminology is important, and the following is to be noted.

Definition 22 Given a security, the yield $y(c)$ of the underlying cash flow

$$c = ((0, -NP_0), (1, ND), \ldots, (n-1, ND), (n, ND + R))$$

is called the *yield to redemption.*

If the security is traded for P_k per unit nominal at time k, then the ratio D/P_k of dividend (coupon) rate and price per unit nominal is called the *running yield* of the security at time k.

For equities the running yield is the analogue with dividend instead of coupon rates.

The price P_k determines the current capital value of the security, and the running yield then expresses the rate at which interest is paid on the capital value.

This distinction of yield to redemption and running yield is related to the notions of interest income and capital gains relevant for taxation. Coupon and dividend payments are considered income, whereas any profit due to different purchase and redemption prices is considered capital gain. The yield to redemption takes into account both income and capital gains (or losses), whereas the running yield only contains the income part.

Example 28 Given a typical 6% security (payable semi-annually) with redemption date three years from now that is currently traded above par at 105%. We calculate the running yield as $6/105 \approx 5.7\%$. The yield to redemption is the solution of

$$\begin{aligned} 0 &= i\text{-}Val_0((0,-105),(0.5,3),(1,3),(1.5,3),(2,3),(2.5,3),(3,103)) \\ &= -105 + 6a^{(2)}_{\overline{3}|i} + (1+i)^{-3}100 \end{aligned}$$

and numerically, we calculate a yield to redemption of $\approx 4.3\%$. Clearly, the difference is due to the capital loss.

10.2 Income tax and capital gains tax

Example 29 The holder of a savings account at 2.5% gross interest is usually liable to *income tax* on savings at a rate of $t_1 = 20\%$, reducing his interest rate to 2% net.

Example 30 An investor who buys equities for C and sell for $S > C$ within a year, pays 40% capital gains tax of $S - C$.

The taxation legislation is complex and not subject of this course. We always place ourselves in situations where we assume to be given tax rates and whether or not an investor and his investment are liable to these taxes. Nevertheless, we need to distinguish the two taxes in certain situations, the most important and straightforward case is fixed-interest securities.

In general terms, *income tax* is applied to interest income that is typically payable regularly. *Capital gains tax* is payable on sale or redemption of equities, securities or, in principle any other financial product. The difference between sales and purchase price is called the *capital gain* , for obvious reasons. If it is positive, tax is applied. If it is negative, no tax is payable. Under certain restrictions, one may offset capital losses L against other taxable capital gains G so as to pay tax only on $G - L$ if $G > L$.

Example 31 If the holder of a fixed-interest security is liable to income tax at rate t_1 and capital gains tax at rate t_2, in principle, and if the fixed-interest security is not exempt from any of these taxes, then the liabilities are as follows.

Then income tax is applied to the coupon payments of ND at times $k = 1, \ldots, n$, at rate D on the nominal value N, reducing the payments to $ND(1 - t_1)$.

If held for the whole term, then the difference of redemption price R and purchase price A is subject to capital gains tax, reducing the redemption proceeds to $R - t_2(R - A)^+$.

If not held for the whole term but sold at time k for P_k per unit nominal, capital gains tax reduces the sales proceeds P_kN to $P_kN - t_2(P_kN - A)^+$.

It can be argued that capital gains tax on $(R - A)^+$ is not quite fair, particularly if purchase and sale are far apart, because of inflation. In fact, there is an adjustment allowing for inflation.

10.3 Inflation adjustments

Example 32 If capital gains tax allows for inflation, then buying a financial product (equities, say) for $\pounds_{1998}100$ in January 1998 (RPI=159.5) and selling it for $\pounds_{2002}180$ in January 2002 (RPI=173.3) entails tax deductions of

$$t_2(180 - 108.65) \qquad \text{since} \qquad \pounds_{1998}100 = \pounds_{1998}100\frac{\pounds_{2002}173.3}{\pounds_{1998}159.5} = \pounds_{2002}108.65.$$

We return to a general inflation index Q. If you can achieve a yield $y(c)$ on cash flow c, you can, in principle, we have seen last lecture that you can make this yield a real yield by adjusting the cash flow to inflation:

Given a discrete cash flow $c = ((t_1, C_1), \ldots, (t_n, C_n))$ and an inflation index Q, we defined the inflation adjustment c^Q of c as

$$c^Q = ((t_1, C_1^Q), \ldots, (t_n, C_n^Q)) = \left(\left(t_1, \frac{Q(t_1)}{Q(0)}C_1\right), \ldots, \left(t_n, \frac{Q(t_n)}{Q(0)}C_n\right)\right).$$

Also, in analogy, given a continuous cash flow $c(t)$, $t \in [0, \infty)$, we define its inflation adjustment $c^Q(t)$, $t \in [0, \infty)$ as

$$c^Q(t) = \frac{Q(t)}{Q(0)}c(t).$$

Proposition 20 *If a cash flow c has yield $y(c)$, then the inflation adjusted cash flow c^Q has real yield $y_R(c^Q) = y(c)$ under inflation Q.*

Proof: We apply the Definition of the real yield equation for the to express $y_R(c^Q)$ as the unique root j of

$$0 = \sum_{k=1}^{\infty}(1+j)^{-t_k}\frac{Q(0)}{Q(t_k)}C_k^Q + \int_0^{\infty}(1+j)^{-t}\frac{Q(0)}{Q(t)}c^Q(t)dt$$

and we see the inflation rate cancel out leaving $0 = NPV(j)$ whose root is the yield $y(c)$. We can also see this from Proposition 18. $\square$

Example 33 An index-linked security is the inflation adjusted cash flow c^Q derived from a fixed-interest security c, i.e.

$$\begin{aligned} c &= \left(\left(\frac{1}{2}, \frac{D}{2}\right), \ldots, \left(\frac{2n-1}{2}, \frac{D}{2}\right), \left(n, \frac{D}{2} + R\right)\right) \\ \Rightarrow \quad c^Q &= \left(\left(\frac{1}{2}, \frac{D}{2}\frac{Q(1/2)}{Q(0)}\right), \ldots, \left(\frac{2n-1}{2}, \frac{D}{2}\frac{Q\left(\frac{2n-1}{2}\right)}{Q(0)}\right), \left(n, \left(\frac{D}{2} + R\right)\frac{Q(n)}{Q(0)}\right)\right). \end{aligned}$$

Typically, $Q(t) = RPI(t - 8/12)$, since at the time of the dividend payment, the current value of the RPI is not known. A delay of 8 months may seem long, but the advantage is, that both parties know the amount well in advance and can plan accordingly.

If after the term of the security, the holder wants to know his real yield on the security, he would use RPI rather than Q, so the inflation adjustment does not guarantee a real yield known in advance, but is usually very close to it.

Lecture 11

Uncertain payments and corporate bonds

So far, we have assumed all in- and outflows as well as interest rates were known. This is rarely the case in practice, and probabilistic models can help to deal with this.

In particular, every company has a (usually small) risk of default that should be taken into account when assessing any corporate investments. This can be dealt probabilistically.

11.1 An example

Example 34 Suppose, you are offered a zero-coupon bond of £100 nominal redeemable at par at time 1. Current market interest rates are 4%, but there is also a 10% risk of default, in which case no redemption payment takes place. What is the fair price?

The present value of the bond is 0 with probability 0.1 and $100(1.04)^{-1}$ with probability 0.9. The weighted average $A = 0.1 * 0 + 0.9 * 100(1.04)^{-1} \approx 86.54$ is a sensible candidate for the fair price.

11.2 Fair premiums and risk under uncertainty

Definition 23 In an interest model δ, the *fair premium* for a random cash flow (of fixed length)

$$C = ((T_1, C_1), \ldots, (T_n, C_n))$$

(typically of benefits $C_j \geq 0$) is the mean value

$$A = E\left(DVal_0(C)\right) = \sum_{j=1}^{n} E\left(C_j v(T_j)\right)$$

where $v(t) = \exp\{-\int_0^t \delta(s)ds\}$ is the discount factor at time t.

Proposition 21 *If the times of a random cash flow are fixed $T_j = t_j$ and only the amounts C_j are random, the fair premium is $A = \sum_{j=1}^{n} E(C_j)v(t_j)$ and depends only on the mean amounts.*

Proof: The deterministic v^{t_j} can be taken out of the expectation. □

Proposition 22 *If the amounts of a random cash flow are fixed $C_j = c_j$ and only the times T_j are random, the fair premium is $A = \sum_{j=1}^{n} c_j E(v(T_j))$ and in the case of constant δ we have $E(v(T_j)) = E(\exp\{-\delta T_j\}) = E(v^{T_j})$ which is the so-called Laplace transform or (moment) generating function.*

Proof: Again, the deterministic c_j can be taken out of the expectation. □

The fair premium is just an average of possible values, i.e. the actual random value of the cash flow C is higher or lower with positive probability each as soon as $DVal_0(C)$ is truly random. In a typical insurance framework when $C_j \geq 0$ represents benefits that the insurer has to pay us under the policy, we will be charged a premium that is higher than the fair premium, since the insurer has (expenses that we neglect and) the risk to bear that we want to get rid of by buying the policy, and that we assess now.

Call A_+ the higher premium that is to be determined. Clearly, the insurer is concerned about his loss $(DVal_0(C) - A_+)^+$, or expected loss $E\left(DVal_0(C) - A_+\right)^+$. In some special cases this can be evaluated, sometimes the quantity under the expectation is squared (so-called squared loss). A simpler quantity is the probability of loss $P(DVal_0(C) > A_+)$. In the next lecture we shall use Tchebychev's inequality to indicate that the insurer's risk of loss gets smaller with an increasing number of policies.

11.3 Uncertain payment

Before discussing pricing issues specific to corporate investments, we introduce some terminology.

Definition 24 An *uncertain cash flow C* is a sequence $(t_j, c_j, B_j)_{j\geq 1}$ with $t_j \in [0, \infty)$, $c_j \in \mathbb{R}$ and $B_j \sim \mathcal{B}(1, p_j)$ Bernoulli random variables with parameters $p_j \in [0, 1]$ and/or a random function $(c, B) : [0, \infty) \to \mathbb{R} \times \{0, 1\}$ such that $B(t) \sim \mathcal{B}(1, p(t))$. p_j and $p(t)$ are interpreted as the probabilities that the corresponding in- or outflow takes place. We make no restrictions on the dependence structure of the B_j and $B(t)$.

Restrictions on the dependence structure of the Bernoulli random variables are not necessary since we shall here only take expectations of linear functionals of these random variables, and these do not depend on the dependence structure.

Uncertain cash flows are special cases of generalised cash flows introduced in Lecture 1. The randomness only enters via payment/no-payment possibilities for any individual in- and outflow.

11.4 Pricing of corporate bonds

Let

$$c = ((1, DN), \ldots, (n-1, DN), (n, DN+N))$$

be a simple fixed-interest security, but issued by a company. The risk of insolvency of the company needs to be added to c: model the insolvency time T as a random variable and define $B_k = 1_{\{T>k\}}$. A corporate bond is then the uncertain cash flow C given by

$$((1, DN, B_1), \ldots, (n-1, DN, B_{n-1}), (n, DN+N, B_n)).$$

Here we assume implicitly that insolvency entails a complete loss of coupon payments and initial capital. In certain situation a refined model may be more adequate.

The main result of this section applies for general cash flows c that are "killed" at a continuous insolvency time T. We denote the killed cash flow by $c_{[0,T]}$.

Proposition 23 *Let $\delta(\cdot)$ be an interest rate model, c a cash flow and T a continuous insolvency time, then*

$$E\left(\delta\text{-}DVal_0\left(c_{[0,T]}\right)\right) = \tilde{\delta}\text{-}DVal_0(c)$$

where $\tilde{\delta} = \delta + \mu$ and $\mu(t) = f_T(t)/\bar{F}_T(t)$.

Proof: First note that

$$\bar{F}_T(t) = P(T > t) = \exp\left\{-\int_0^t \mu(s)ds\right\}$$

as is easily verified by taking logarithms and differentiation. $p(t) = P(T > t)$ is the parameter of $B(t)$, and $p_j = P(T > t_j)$ of B_j. Therefore,

$$E(\delta\text{-}DVal_0(C)) = \sum_{j=1}^{\infty} c_j e^{-\int_0^{t_j} \mu(s)+\delta(s)ds} + \int_0^{\infty} c(t)e^{-\int_0^t \mu(s)+\delta(s)ds}dt$$

as required. □

The important special case is when δ and μ are constant and c is a corporate bond influenced by the insolvency time T.

Corollary 7 *Given a constant δ model, the fair price for a corporate bond C with insolvency time $T \sim Exp(\mu)$ is*

$$A = E(\delta - DVal_0(C)) = \tilde{\delta} - DVal_0(c)$$

where $\tilde{\delta} = \delta + \mu$.

Often, problems arise in a discretised way. Remember that a geometric random variable with parameter p can be thought of as the first 0 in a series of Bernoulli 0-1 trials with success (1) probability p.

Proposition 24 *Let c be a discrete cash flow with $t_k \in \mathbb{N}$ for all $k = 1, \ldots, n$, $T \sim$ geom(p), i.e. $P(T = k) = p^{k-1}(1-p)$, $k = 1, 2, \ldots$. Then in the constant i model,*

$$E\left(i\text{-}Val_0(c_{[0,T)})\right) = j\text{-}Val_0(c)$$

where $j = (1 + i - p)/p$.

Proof: $P(T > k) = p^k$. Therefore

$$E\left(i\text{-}Val_0(c_{[0,T)})\right) = \sum_{k=1}^{n} p^{t_k} c_k (1+i)^{-t_k}.$$

$\square$

Lecture 12

Life insurance: the single decrement model

'Single decrement' means that there is only one change of state, from 'alive' to 'dead'. More general 'multiple decrement models' include e.g. illness.

12.1 Uncertain cash flows in life insurance

Assume that a certain cash flow c is restricted to some (residual) lifetime T. This was our situation for corporate bonds and related cash flows stopping at the insolvency time T of a company. Here we think of T as the time to death of a human being (life).

Example 35 (Pure endowment) A *pure endowment* provides just one unit payment $(n, 1)$ if and when the person insured lives at time n, i.e. if $T > n$. Clearly, the fair premium in an interest model $\delta(\cdot)$ is $A = v(n)P(T > n)$.

More typically, the cash flow c might be a pension payable from now or from some future date. To price these, we can reformulate Proposition 23 in Lecture 11.

Proposition 25 *Let $\delta(\cdot)$ be an interest rate model, c a cash flow and T the time of death of a life, modelled by a continuous random variable, $c_{[0,T]}$ the cash flow c restricted to $[0, T]$, then the fair price for $c_{[0,T]}$ is*

$$E\left(\delta\text{-}DVal_0\left(c_{[0,T]}\right)\right) = \tilde{\delta}\text{-}DVal_0(c)$$

where $\tilde{\delta} = \delta + \mu_T$ and $\mu_T(t) = f_T(t)/\bar{F}_T(t)$.

In fact, $\mu_T(t)$, $t \geq 0$, is of importance in the sequel. It has a name:

Definition 25 For any continuous random variable T in $[0, \infty)$ modelling a lifetime, the function $\mu(t) = f_T(t)/\bar{F}_T(t)$ is called the force of mortality.

We will give a motivation for this name in the next section. We conclude this section introducing an important class of examples, life annuities.

Example 36 (Life annuities) An *elementary life annuity* for a life with residual life time T, is the cash flow $C = c_{[0,T]}$ obtained by restricting the perpetuity $c = ((n,1))_{n\in\mathbb{N}}$ to $[0,T]$, i.e.

$$c_{[0,T]} = ((1,1),(2,1),\ldots,([T],1))$$

where $[T]$ denotes the integer part of T, i.e. $[T] \leq T < [T]+1$.

Analogously, *pthly* and *continuously payable life annuities* are the restricted $p$thly and continuously payable perpetuities.

Example 37 Given a (residual) life time $T \sim Exp(\mu)$, the fair price of an elementary life annuity $c_{[0,T]}$ in a constant δ interest rate model is

$$A = (\delta+\mu)\text{-}DVal_0(c) = \frac{1}{e^{\delta+\mu}-1} = \frac{p}{1+i-p}$$

where $i = e^{\delta} - 1$ and $p = e^{-\mu} = P(T > 1)$.

12.2 Conditional probabilities and the force of mortality

Definition 26 Given a random variable T and a set $B \subset \mathbb{R}$ such that $P(T \in B) > 0$, then the conditional distribution of T given $\{T \in B\}$ is defined for $A \subset \mathbb{R}$ by

$$P(T \in A|T \in B) = \frac{P(T \in A, T \in B)}{P(T \in B)} = \frac{P(T \in A \cap B)}{P(T \in B)}.$$

We still think of T as a life time. Conditional probabilities arise naturally in questions like the following.

Example 38 Given that somebody reaches his 65th birthday, what is the probability that he survives his 80th birthday:

$$P(T > 80|T > 65) = P(T > 80)/P(T > 65).$$

Note that the conditional probability exceeds the unconditional $P(T > 80)$. Intuitively this is because the older you get, the more likely it is that you survive your 80th birthday.

Example 39 You would expect that the older you get, the more likely it is that you die within the following year, month, day etc. This is not reflected in the density f_T of T. f_T is usually even assumed decreasing. In fact, the phrase "the older you get" indicates conditioning on survival, and the probabilities that we expected to increase are

$$P(T \leq t+\varepsilon|T > t), \qquad t \geq 0,$$

for $\varepsilon = 1$, $\varepsilon = 1/12$, $\varepsilon = 1/365$ etc., respectively.

Dealing with several values of ε is a bit inconvenient, but

Proposition 26 *Given a continuous random variable $T \in [0,\infty)$, we have*

$$\frac{1}{\varepsilon}P(T \leq t+\varepsilon|T > t) \to \mu_T(t), \qquad \text{as } \varepsilon \to 0$$

Proof: Just note that

$$P(T \leq t+\varepsilon|T>t) = \frac{P(t<T\leq t+\varepsilon)}{\varepsilon P(T>t)} = \frac{F_T(t+\varepsilon)-F_T(t)}{\varepsilon \bar{F}_T(t)} \to \frac{f_T(t)}{\bar{F}_T(t)} = \mu_T(t).$$

□

The force of mortality expresses your infinitesimal likelihood to die provided you are still alive. It therefore is a measure of your risk to die now.

12.3 The curtate future lifetime

As many cash flows involve payments only on a discrete lattice, integer times, say, it is useful to discretise lifetime distributions.

Definition 27 Given a life time random variable T, we call $K = [T]$ the associated *curtate lifetime.*

K is the number of complete time units (years) lived.

Example 40 If $T \sim Exp(\mu)$, we have $K = [T] \sim geom(e^{-\mu})$. To see this, denote $p = e^{-\mu}$ and note that

$$P(K=k) = P(k \leq T < k+1) = \int_k^{k+1} \mu e^{-\mu t} dt = e^{-k\mu} - e^{-(k+1)\mu} = p^k(1-p).$$

This motivates the expression in terms of p in Example 37.

More generally, we can have μ and p depend on time:

Proposition 27 *The distribution of K is given by $P(K=k) = P(k \leq T < k+1)$. K can be seen as the first 0 in a series of independent Bernoulli trials with varying success (1) probabilities*

$$p_k = P(T \geq k+1|T \geq k) = \exp\left\{-\int_k^{k+1} \mu_T(s)ds\right\},$$

the latter provided T is a continuous random variable.

Proof: The first statement is clear from the definition of $[\cdot]$. We further express the probabilities as

$$\begin{aligned} P(K=k) &= P(T \geq k) - P(T \geq k+1) = P(T \geq k)(1 - P(T \geq k+1|T \geq k)) \\ &= P(T \geq k)(1-p_k) = p_0 \ldots p_{k-1}(1-p_k) \end{aligned}$$

by induction. Clearly, this is also the probability that the first 0 in a series of independent Bernoulli trials with varying success probabilities p_j occurs at k.

In the case of a continuous T we use

$$P(T > t) = \exp\left\{-\int_0^k \mu_T(s)ds\right\}$$

from the proof of Proposition 23, and we further calculate

$$P(T \geq k+1 | T \geq k) = \frac{P(T \geq k+1)}{P(T \geq k)} = \exp\left\{-\int_k^{k+1} \mu_T(s)ds\right\}.$$

□

12.4 Insurance types and examples

In the sequel we assume given a life with time to death T, and $K = [T]$, and a constant δ interest model.

We have seen pure endowments in Example 35 and life annuities in Examples 36 and 37. Three more types are important.

Example 41 (Whole life insurance) Pay one unit at the end of the year of death, i.e. at time $K+1$. The random discounted value at time 0 is $Z = e^{-\delta(K+1)}$ and $A(\delta) = E(Z) = E(e^{-\delta(K+1)})$ is the fair premium. We can also calculate $E(Z^m) = E(e^{-m\delta(K+1)}) = A(m\delta)$ and $Var(Z) = A(2\delta) - A^2(\delta)$.

Example 42 (Term insurance) Pay one unit at the end of the year of death if death occurs within n years. No payment is made if the life survives n years.

Example 43 (Endowments) Formally, this is the sum of a term insurance and a pure endowment: pay one unit at the end of the year of death if death occurs within n years, otherwise pay one unit after n years.

There has been a lot of effort to find suitable parametric families of lifetime distributions that have simple survival functions. The most elementary and most notable ones are

Example 44 (Gompertz-Makeham) $\mu_T(t) = A + Bc^t$ for $A > 0$, $B > 0$, $c > 0$, which implies

$$\bar{F}_T(t) = \exp\{-At - m(c^t - 1)\}$$

where $m = B/\log(c)$

Example 45 (Weibull) $\mu_T(t) = kt^n$ *for* $k > 0$, $n > 0$, *which implies*

$$\bar{F}_T(t) = \exp\left\{-\frac{k}{n+1}t^{n+1}\right\}.$$

In practice, one does not restrict to two or three parameters, but estimates laws of $K = [T]$ essentially among all distributions on $\mathbb{N}$ or $\{0, 1, \ldots, 109\}$ from past experience. This law of K is the main information given on lifetables.

Lecture 13

Life insurance: premium calculation

13.1 Residual lifetime distributions

Actuaries often work with large populations of different ages. It is therefore convenient to introduce notation taking account of age.

Definition 28 (Family of residual lifetime distributions) For a given (full) lifetime T we define a family of random variables (actually probability distributions)

$$P(T_x \in A) = P(T - x \in A | T > x) \qquad \text{or} \qquad \bar{F}_{T_x}(t) = \frac{\bar{F}_T(t+x)}{\bar{F}_T(x)}$$

for $A \subset \mathbb{R}_+$, $t \in \mathbb{R}_+$, $x \in [0, \omega)$ where $\omega = \inf\{t \geq 0 : \bar{F}_T(t) = 0\}$ is the maximal age possible under T. T_x is called the residual lifetime of a life aged x.

Proposition 28 *Any family of residual lifetime distributions $(T_x)_{x \in [0,\omega)}$ in the sense of Definition 28 has the consistency property*

$$P(T_{x+y} \in A) = P(T_x - y \in A | T_x > y)$$

for all $x, y \in \mathbb{R}_+$ such that $x + y < \omega$. Vice versa, any consistent family of residual lifetime distributions is uniquely determined by a full lifetime distribution of $T = T_0$.

Proof: The distribution function, and hence the survival function uniquely determines the distribution of a random variable, it therefore suffices to note that

$$P(T_x - y > t | T_x > y) = \frac{\bar{F}_{T_x}(y+t)}{\bar{F}_{T_x}(y)} = \frac{\frac{\bar{F}_T(x+y+t)}{\bar{F}_T(x)}}{\frac{\bar{F}_T(x+y)}{\bar{F}_T(x)}} = \frac{\bar{F}_T(x+y+t)}{\bar{F}_T(x+y)} = \bar{F}_{T_{x+y}}(t).$$

The second statement is obvious as the consistency condition contains the definition of the law of T_y for $x = 0$. □

The law of T_x can be used both as the residual lifetime distribution of a life aged x now and as the conditional residual lifetime distribution beyond x of a life that is younger now given it survives to age x. This is often useful in applications.

13.2 Actuarial notation for life products

We recall that $K = [T]$ denotes the total number of completed future years of a life. Analogously, we denote $K_x = [T_x]$ for a life aged x now. Actuaries have extensive notation related to lifetimes. We only introduce some key notation here.

Definition 29 Actuaries use the following shorthand for lifetime distributions

$$_tp_x = \bar{F}_{T_x}(t) = P(T_x > t) \qquad \text{and} \qquad _tq_x = 1 - {_tp_x} = F_{T_x}(t) = P(T_x \leq t)$$

where a pre-index $t = 1$ is usually suppressed: $p_x = {_1p_x}$ and $q_x = {_1q_x}$. The force of mortality is also denoted

$$\mu_{t+x} = \mu_{T_x}(t).$$

Note that for $t = k \in \mathbb{N}$, the symbols also denote the survival and distribution function of the curtate lifetime K. q_x is particularly important being the probability of dying within a year for a life aged x.

Example 46 In Proposition 27 we used notation $p_k = P(T \geq k+1 | T \geq k) = P(T_k > 1) = P(K_k > 1)$ consistent with the definition here to interpret the (curtate) lifetime as the first failure in a series of Bernoulli experiments. The p_k (or the $q_k = 1 - p_k$), $k \in \mathbb{N}$ determine the law of K. This observation is crucial when reading lifetables that provide estimates of just q_k, $k \in \mathbb{N}$.

Further notation is particularly useful to represent and relate the fair prices of main insurance products.

Example 47 (Whole life assurance) Given a constant interest rate model, the fair premium of a whole life insurance is denoted

$$A_x = A_x(\delta) = E(\exp\{-\delta(K_x + 1)\}) = \sum_{k=0}^{\infty} v^{k+1} {_kp_x} q_{x+k}$$

where the underlying force of interest δ is often suppressed.

Example 48 (Term assurance, pure endowment and endowment) The fair premium of a term insurance is denoted by

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {_kp_x} q_{x+k}.$$

The superscript 1 above the x indicates that 1 is only paid in case of death within the period of n years.

The fair premium of a pure endowment is denoted by $A_{x:\overset{1}{\overline{n}|}} = v^n {_np_x}$. Here the superscript 1 indicates that 1 is only paid in case of survival of the period of n years.

The fair premium of an endowment is denoted by $A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$, where we could have put a 1 above both x and n, but this is omitted being the default, like in previous symbols.

13.3 Lifetables

Assume that there is a (curtate) lifetime distribution K that we want to estimate (and associated K_x). We know how to express it in terms of one-year death probabilities q_x, $x = 0, 1, 2, \ldots$. The naive way to estimate q_x is the following.

Proposition 29 *Given a sample of n people observed between age x and $x+1$, record $B_j = 1$ ($B_j = 0$) if person j died (survived), $j = 1, \ldots, n$. Then $B_j \sim \mathcal{B}(1, q_x)$ can be assumed independent random variables and*

$$\hat{q}_x = \hat{q}_x(n) = \frac{1}{n}\sum_{j=1}^{n} B_j \to q_x$$

in probability, as n tends to infinity.

Proof: $B_j \sim \mathcal{B}(1, q_x)$ is clear since q_x is the one year death probability of any person aged x. The convergence in probability follows from the weak law of large numbers. □

For given data, $\hat{q}_x$ is an estimate of q_x. Life tables show such estimates rather than any true values behind, although they write q_x rather than $\hat{q}_x$, for historical reasons, since such 'estimates' have been used much longer than the stochastic approach behind. An extract from a life table is reproduced in Appendix A.

In practice, there are some complications and ways to address these that we only indicate here:

- incomplete observations (people entering an insurance contract between ages x and $x+1$, or whose contract reaches the end of its term);
- small sample sizes for high x, e.g. $x = 105$; a procedure called 'graduation' averages with neighbouring values and thus also completes the picture up to a maximal age;
- the more data one considers, the older they are; since mortality changes with time, estimates are likely not to be up to date; a procedure called 'extrapolation' projects forward the development during past years;
- people can be classified into groups with significantly different mortality: male/female, smoker/nonsmoker, job groups, state of health, type of insurance chosen etc; there are specific tables for the more important combinations of these;
- insurance contracts can often only be made when in a good state of health; this decreases mortality significantly, and it is customary to take account of this for two initial years, denoting decreased death probabilities by $q_{[x]}$ and $q_{[x]+1}$ respectively. With these included, a life table has three columns and the relevant entries for any given situation are the age row and the q_{x+2+n}, $n \in \mathbb{N}$, in the last column.

Example 49 The premium for a 4-year temporary life assurance of a life aged 55 assuming an interest rate $i = 4\%$ is

$$A^{1}_{55:\overline{4}|} = vq_{55} + v^2 p_{55} q_{56} + v^3 p_{55} p_{56} q_{57} + v^4 p_{55} p_{56} p_{57} q_{58},$$

where we recall that $p_x = 1 - q_x$.

Given a sum assured of $N = £10,000$ and without taking account of a good initial state of health we read off the life table given in Appendix A and calculate

$$NA^{\,1}_{55:\overline{4}|} \approx 356.26.$$

Taking account of a good initial state of health, i.e. using $q_{[55]}$ and $q_{[55]+1}$ instead of q_{55} and q_{56}, we obtain from the same table

$$NA^{\,1}_{[55]:\overline{4}|} \approx 293.13.$$

13.4 Life annuities

Example 50 Given a constant i interest model, the fair premium of an ordinary (respectively temporary) life annuity for a life aged x is given by

$$a_x = \sum_{k=1}^{\infty} v^k {}_kp_x \quad \text{respectively} \quad a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k {}_kp_x.$$

For an ordinary (respectively temporary) life annuity-due, an additional certain payment at time 0 is made:

$$\ddot{a}_x = 1 + a_x \quad \text{respectively} \quad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n}|}.$$

Life annuities occur, in particular when life product premiums are multiple rather than single premiums: in the simplest case level advance premiums are paid until death. The cash flow of premium payments is therefore a life annuity. We shall see in the next lecture that this observation allows to calculate the amount of fair level premiums.

13.5 Multiple premiums

Definition 30 Given a constant i interest model, let C be the cash flow of insurance benefits, the annual fair level premium of C is defined to be

$$P_x = \frac{E(DVal_0(C))}{\ddot{a}_x}.$$

Proposition 30 *In the setting of Definition 30, the expected dicounted benefits equal the expected discounted premium payments.*

Proof: The expected discounted value of premium payments $((0, P_x), \ldots, (K_x, P_x))$ is $P_x \ddot{a}_x = E(DVal_0(C))$. □

Example 51 We calculate the fair annual premium in Exercise 49. Since there should not be any premium payments beyond the term of the assurance, the premium payments are a term life annuity-due $((0, 1), (1, 1), (2, 1), (3, 1))$ restricted to the lifetime K_{55}. First

$$\ddot{a}_{[55]:\overline{4}|} = 1 + vp_{[55]} + v^2 p_{[55]}p_{[55]+1} + v^3 p_{[55]}p_{[55]+1}p_{57} \approx 3.74143$$

and the annual premium is therefore calculated from the life table in Appendix A as

$$P = \frac{NA^{\,1}_{[55]:\overline{4}|}}{\ddot{a}_{[55]:\overline{4}|}} = 78.35.$$

Lecture 14

Pricing of equity shares and risk

14.1 Pricing of equity shares

The evolution of share prices largely depends on expectations in the future profitability of the company's business. One method of explaining share prices is by discounting future dividends:

Definition 31 Given a probabilistic (or deterministic) model for the future dividend payments $(D_k)_{k\geq 1}$ of a share and an interest rate model $\delta(\cdot)$, the *discounted dividend price* of the share is given by

$$P = \sum_{k\geq 1} v(k)E(D_k).$$

The definition assumes that dividends are paid once per time unit. Generalisations are straightforward. It is implicitly assumed that the random variables D_k include the no-payment possibility due to insolvency (or other financial restrictions).

If a share is held in perpetuity, this is clearly the fair price of this share.

Example 52 An equity share is expected to pay constant dividend D forever with no risk of insolvency. Interest rates are expected to be constant i. Then the discounted dividend price for the share is the price of the perpetuity $(k, D)_{k\geq 0}$

$$P = \sum_{k\geq 1}(1+i)^{-k}D = \frac{D}{1+i}\frac{1}{1-(1+i)^{-1}} = \frac{D}{i}.$$

Other popular models assume

$$D_k = d_k = d_0(1+j)^k,$$

deterministic, for some growth rate j. These can be thought of as coming from stochastic models such as

$$D_k = d_0\prod_{m=1}^{k}(1+A_m),$$

where the A_m, $m \geq 1$, are independent and identically distributed random growth rates with $E(A_m) = j$, because the product rule for expectations of independent random variables ($E(XY) = E(X)E(Y)$) readily gives

$$E(D_k) = d_0 E\left(\prod_{m=1}^{k}(1 + A_m)\right) = d_0 \prod_{m=1}^{k} E(1 + A_m) = d_0(1 + j)^k.$$

In this section, more obviously than in many previous applications, P is only a *model* price. In practice, often the prices are given (determined by the market). Given some interest rate assumptions i and the last dividend payment d_0, you can calculate j and buy if you believe that a rate of dividend increase of j is realistic.

But, be careful: d_0 may be low for very different reasons. Dividends are paid from the profit of the company, so possible reasons are

- The company has no profit in the year. You might consider this a bad perspective for future growth.
- Decisions to invest profit into expansions rather than paying out to shareholders, have been taken by the Assembly of Shareholders. This can mean, that a new factory is built from the profits, and this can lead to substantially higher dividends in future years when profit may be passed on to shareholders, again.

14.2 Individual risk models

Definition 32 The *individual risk model*

$$(B_j, X_j)_{1 \leq j \leq n}$$

describes a portfolio of $n \in \mathbb{N}$ insurance policies (over a given fixed short time period, ignore any effects of interest). For each policy $j = 1, \ldots, n$ the number of claims $B_j \in \{0, 1\}$ and (if $B_j = 1$) the amount of the claim $X_j \in (0, \infty)$ are random variables, independent but not necessarily identically distributed for different j.

We denote by $Y_j = B_j X_j$ the payment under the jth policy and by $S = Y_1 + \ldots + Y_n$ the aggregate total claim amount of all policies.

Net premiums for the policies are $E(Y_j)$. We will now study the risk associated, more precisely, the insurer's risk that the benefits are higher than the premium, individually, or for a portfolio.

14.3 Pooling reduces risk

Assume an individual risk model with n policies. The random total claim amount S must be ensured by premium payments A, say, to be determined. Usually, the fair premium $E(S)$ leaves too much risk to the insurer. E.g. the loss probabilities $P(S > A)$ is usually too high. The following result suggests to set a higher premium to ensure a low loss probability.

Proposition 31 *Given a random variable Y_1 with mean μ and variance σ^2, representing the benefits from an insurance policy, we have*

$$P\left(Y_1 \geq \mu + \frac{\sigma}{\sqrt{\delta}}\right) \leq \delta,$$

and $A_1(\delta) = \mu + \sigma/\sqrt{\delta}$ is the premium to be charged to achieve a loss probability below δ.

Given n independent and identically distributed Y_j from n independent policies, we obtain

$$P\left(\sum_{j=1}^{n} Y_j \geq n\left(\mu + \frac{\sigma}{\sqrt{n\delta}}\right)\right) \leq \delta.$$

i.e. $A_n(\delta) = \mu + \sigma/\sqrt{n\delta}$ suffices if the risk of n policies is pooled.

Proof: The statements follow as consequences of Tchebychev's inequality:

$$P\left(\sum_{j=1}^{n} Y_j \geq n\left(\mu + \frac{\sigma}{\sqrt{n\delta}}\right)\right) \leq P\left(\left|\frac{1}{n}\sum_{j=1}^{n} Y_j - \mu\right| \geq \frac{\sigma}{\sqrt{n\delta}}\right) \leq \frac{\sigma^2}{n\frac{\sigma^2}{n\delta}} = \delta.$$

□

The estimates used in this proposition are rather weak, and the premiums suggested require some modifications in practice, but adding a multiple of the standard deviation is one important method, also since often the variance, and hence the standard deviation, can be easily calculated. However, for large n, so-called safety loadings $A_n(\delta) - \mu$ proportional to $n^{-1/2}$ are of the right order.

The important observation in this result is that the premiums $A_n(\delta)$ decrease with n. This means, that the more policies an insurer can sell the smaller gets the (relative) risk, allowing him to reduce the premium. The proposition indicates this for identical policies, but in fact, this is a general rule about risk. You can also test it on a personal level: often insurance policies are sold in packages. You can insure your house, its contents, personal liability, travelling etc. in one policy. This is much cheaper than insuring every single item or liability separately.

Another question is, whether the probability of loss (loss to the insurer) is the right quantity to inspect when assessing risk. One can argue that it is more appropriate to take into account the size of the loss, if loss occurs, e.g.

$$E(loss) = E((S_n - nA_n)^+)$$

where $S_n = Y_1 + \ldots + Y_n$ are the total benefits paid, and A_n is the premium per policy. Or

$$E(loss^2) = E(((S_n - nA_n)^+)2)$$

to put more weight on high losses. If $A_n = A_n^0 = E(Y_n)$ was the net premium, this expression would be close to the variance of S_n (just remove the positive part $(\cdot)^+$). This can be taken as a further motivation to look at the variance as a (first) measure of risk.

Lecture 15

Premium principles

15.1 Examples from life insurance

Example 53 Consider a 10-year temporary assurance for a life aged 40, with sum assured C. Consider a constant interest rate of 4% and a residual lifetime distribution that is uniform with terminal age 100. Then we calculate

$$A^{\,1}_{40:\overline{10}|} = \frac{1}{60}v + \frac{1}{60}v^2 + \ldots + \frac{1}{60}v^{10} = \frac{1}{60}a_{\overline{10}|} = 0.1352$$

as the net single premium.

To calculate the net annual premium, consider net single premiums for pure endowment

$$A_{40:\overset{1}{\overline{10}|}} = \frac{50}{60}v^{10} = 0.5630,$$

and wholelife assurance

$$A_{40:\overline{10}|} = A^{\,1}_{40:\overline{10}|} + A_{40:\overset{1}{\overline{10}|}} = 0.6982.$$

The latter is useful to calculate (using the formula from the proposition below)

$$\ddot{a}_{40:\overline{10}|} = \frac{1 - A_{40:\overline{10}|}}{d} = 7.8476$$

and hence the net annual premium for the temporary assurance:

$$P_0 = C\frac{A^{\,1}_{40:\overline{10}|}}{\ddot{a}_{40:\overline{10}|}} = 0.0172C.$$

In the example, we applied

Proposition 32 *For all $x < \omega$ and $n \geq 1$, we have*

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d}.$$

Proof: We start on the right hand side

$$\begin{aligned}
\frac{1-A_{x:\overline{n}|}}{d} &= \frac{1}{d}\left(1-\sum_{k=1}^{n-1} v^k\ {}_{k-1}p_x q_{x+k-1} - v^n\ {}_{n-1}p_x\right) \\
&= \frac{1}{1-v}\left(1-\sum_{k=1}^{n-1} v^k\ {}_{k-1}p_x + \sum_{k=0}^{n-1} v^k\ {}_{k}p_x - v^n\ {}_{n-1}p_x\right) \\
&= \frac{1}{1-v}\left(\sum_{k=0}^{n-1} v^k\ {}_{k}p_x - v\sum_{j=0}^{n-1} v^j\ {}_{j}p_x\right) \\
&= \sum_{j=0}^{n-1} v^j\ {}_{j}p_x = \ddot{a}_{x:\overline{n}|}.
\end{aligned}$$

□

This proposition is the analogue of the formula

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}.$$

This can be used for an alternative proof using more probabilistic arguments

$$\frac{1-A_{x:\overline{n}|}}{d} = E\left(\frac{1-v^{\min(K_x+1,n)}}{d}\right) = E\left(\ddot{a}_{\overline{\min(K_x+1,n)}|}\right) = \ddot{a}_{x:\overline{n}|}.$$

In practice, insurers should use a safety loading that reflects the assumed risk. This can be done using utility functions $u : \mathbb{R} \to \mathbb{R}$ that satisfy $u'(x) > 0$ and $u''(x) < 0$ for all $x \in \mathbb{R}$ (or $x \in (a, b)$). $u(x)$ is meant to measure the utility that the insurer has of a monetary unit x. $x < 0$ correspond to negative capital, and high negative values should be of strongly negative utility. A popular choice is

$$u(x) = \frac{1}{a}(1-e^{-ax}).$$

The net premium principle that expected discounted benefits should equal the expected discounted premium, is now replaced by

$$E(u(-L)) = u(0)$$

where L is the loss of the insurer. Premiums are chosen such that the expected utility is zero.

Example 54 In the previous example, if an annual premium of P is charged, the loss of the insurer is given by

$$L = \begin{cases} Cv^{K_{40}+1} - P\ddot{a}_{\overline{K_{40}+1}|} & \text{for } K_{40} = 0,1,\ldots,9, \\ -P\ddot{a}_{\overline{10}|} & \text{for } K_{40} \geq 10. \end{cases}$$

For a given choice of a to be discussed below, P is now determined such that $E(u(-L)) = u(0) = 0$. This is equivalent to

$$E(e^{aL}) = 1$$

i.e.

$$\frac{1}{60}\sum_{k=0}^{9} \exp\left\{aCv^{k+1} - aP\ddot{a}_{\overline{k+1|}}\right\} + \frac{5}{6}\exp\left\{-aP\ddot{a}_{\overline{10|}}\right\} = 1.$$

For $a = 10^{-6}$, this gives for various sums assured C:

Sum assured C	P	Percent of P_0
100,000	1,790	104%
500,000	10,600	123%
1,000,000	26,400	153%
5,000,000	1,073,600	1248%

where P_0 is the net annual premium calculated in the previous exercise. We say that for a sum assured of 100,000, the insurer takes a *safety loading* of 4%, whereas for a sum assured of 5,000,000, a safety loading of 1148% would be taken. This, at first sight, unrealistic figure can be understood as follows. The insurer is usually dealing with sums assured not exceeding $a^{-1} = 1,000,000$. Taking on to assure 5,000,000, puts him at considerable risk, which he does not want to do. In practice, rather than charging a premium as suggested, the insurer would send the customer to an insurer with a lower a-value, or arrange for *reinsurance* at such an insurer dealing with higher sums assured. Note that the above formulas are such that an insurer with $a = 10^{-7}$ would insure a sum assured of 5,000,000 at 23% safety loading.

15.2 Some elements of the theory of premium principles

In the sequel, we think of random variables S as the total claim amount from an insurance policy or a portfolio of policies.

Definition 33 A *premium principle* is any rule H that assigns with every random variable S a real number or ∞, denoted $H(S)$. $H(S) - E(S)$ is called *safety loading.* If $H(S) = \infty$, the risk represented by S is called *uninsurable.*

This vague definition reflects a multitude of possibilities.

Example 55 (Net premium principle) The net premium from Definition 23, is the special case $H = E$, i.e. $H(S) = E(S)$.

Example 56 (Expected value principle) $H(S) = (1+\lambda)E(S)$ suggests a safety loading proportional to $E(S)$.

Example 57 (Variance principle) $H(S) = E(S) + \alpha Var(S)$ suggests a safety loading proportional to $Var(S)$.

Example 58 (Standard deviation principle) $H(S) = E(S) + \beta\sqrt{Var(S)}$ suggests a safety loading proportional to the standard deviation of S as in Proposition 31.

Example 59 (Percentile principle) $H(S) = \min\{A|P(S > A) \leq \delta\}$ suggests the minimal premium that bounds the probability of loss by a given level. Proposition 31 does this in an approximate way: the level is not exceeded but the premium is not minimal in this property.

Example 60 (Exponential principle) $H(S) = \log(E(\exp\{aS\}))/a$.

There are other and more general principles using utility functions that we do not discuss here. The exponential principle plays an important role. All principles have advantages and disadvantages.

Some desirable properties that we would like a premium principle H to have

1. Nonnegative safety loading: $H(S) \geq E(S)$ for all S.
2. No ripoff: $P(H(S) \leq S) > 0$.
3. Consistency: $H(S + c) = H(S) + c$.
4. Additivity: If S_1 and S_2 are *independent*, then $H(S_1 + S_2) = H(S_1) + H(S_2)$.

One can then establish the following table

Property	Ex. 55	Ex. 56	Ex. 57	Ex. 58	Ex. 59	Ex. 60
1	yes	no	yes	yes	no	yes
2	yes	no	no	no	yes	yes
3	yes	no	yes	yes	yes	yes
4	yes	yes	yes	no	no	yes

Under certain assumptions, it can be shown that only the net premium principle and the exponential principle have all these properties. Since the net premium principle does not take account of any risk, this makes a strong case for the exponential principle.

Lecture 16

Summary: it's all about Equations of Value

16.1 Summary

Roughly, one can say that the course so far has consisted of three parts. Some of the material mentioned below was introduced on the Assignment sheets as straightforward extensions of lecture material.

16.1.1 Basic notions used throughout the course

The first third (lectures 1,2,3 and parts of 4,6) introduced into

1. Cash flows (discrete cash flows, continuous cash flows, mixed cash flows, generalised random cash flows, modelling with cash flows)

2. Compound interest theory (simple interest, time-dependent compound interest, constant interest rates, simple discount)

3. Valuation of cash flows (accumulated values, discounted values, time values)

4. Annuities (ordinary, $p$thly and continuously payable, perpetuities, deferred annuities and annuities-due)

5. Yields of cash flows (definition and existence of yields)

Expressed differently, the following notation is the most important.

$$c = (t_n, c_n)_{n\geq 0}, \quad c = (c(t))_{t\in\mathbb{R}_+}, \quad C = (T_n, C_n)_{n\geq 0}$$
$$\delta(t),\ v(t), \quad i, v, d, \delta, i^{(p)}, d^{(p)}$$
$$DVal_t(c) + AVal_t(c) = Val_t(c) \qquad NPV(i)$$
$$a_{\overline{n}|}, s_{\overline{n}|}, a^{(p)}_{\overline{n}|}, \ddot{a}_{\overline{n}|}, {}_m|a_{\overline{n}|}, \bar{a}_{\overline{n}|}, a_{\overline{\infty}|}$$
$$y(c)$$

16.1.2 Deterministic applications

The second third (lectures 5,7-10 and parts of 4,6) concerned applications without uncertainty. The following list regroups the main topics.

1. Mortgages and loans (repayment schemes, prospective/retrospective method for outstanding capital, interest/capital repayment components, APR, flat rate)

2. Fixed interest securities (valuation, running yield and yield to redemption, income and capital gains tax)

3. Investment projects (profitability, yields, discounted and undiscounted payback period, comparison of investments, cross-over rates)

4. Funds (Money-weighted rate of return, Time-weighted rate of return, funds as investment)

5. Inflation (RPI index, inflation rate models, real yields, inflation-adjusted payments, capital gains tax taking into account inflation)

16.1.3 Applications with uncertaincy

The last third (lectures 11-15) concerned models with some random ingredients.

1. Corporate bonds (Insolvency time, general pricing formula, pricing under exponential or geometric insolvency time)

2. Discounted dividend model

3. Single decrement model (lifetime distributions, curtate lifetime, lifetables, general valuation, insurance products, single and multiple premiums)

4. Risk (Variance of random values, loss probabilities, fair prices and other premium principles, pooling of risk)

16.2 Equations of value

The most important method throughout the course is via *equations of value*

$$i\text{-}Val_t(c) = X \qquad \text{or} \qquad NPV(i) = X$$

where the interest rate (or yield) i, the time t, the value X or some unknown quantity in the specification of the cash flow c is to be determined. Virtually everything relies on this.

Example 61 (Accumulated and discounted values) X unknown, often t end of term or $t = 0$ (respectively).

Example 62 (Yields, real yields, cross-over rates, APR, MWRR) i unknown, $X = 0$; ...

Example 63 (Discounted payback periods) Here actually $AVal_t(c) = 0$, t unknown

Example 64 (Capital gains tax) Purchase price P of a security unknown given the yield i, say net coupons c, redemption proceeds R:

$$P = NPV(c) + NPV((n, R - (R - P)t_2))$$

Example 65 (Fair prices under uncertainty) Here actually $E(NPV(C)) = X$, usually X unknown

Example 66 (Multiple premiums) Expected discounted benefits equal expected discounted level premium payments; level premium amount P unknown.

Always ask yourself: What is the cash flow? What is the equation of value?

16.3 Examination

The exam paper will consist of 6 questions (not 8 as in most other papers you are going to sit). Questions will be marked out of 25 as usual, so you should aim for 4 questions as usual, the usual marking scheme applies. This means, your amount choice is reduced. This is a concession to the Institute of Actuaries who do not allow any choice in their exams.

You will require a calculator. Please check the University regulations in good time, because you may have to familiarise yourselves with more basic calculators than your own. You will receive letters from the Examiners where this issue is clarified.

The style of questions is not like assignment questions. There is usually a bookwork part asking you to define certain notions or reproduce or reprove important formulas, and checking your understanding of these. A question may cover more than one lecture or more than one of the topics above. Three questions concern MT and three questions HT, but as HT builds on MT they may include explicitly some MT material.

16.4 Hilary Term

Peter Clark who taught classes this term, will take over teaching most lectures next term.

The following topics from the Institute of Actuaries 102 paper remain:

1. Investments: risk characteristics
2. Investments: stability (reaction on small changes in interest rates)
3. How to determine the market interest rate model: term structure of interest rates
4. Stochastic interest rate models

5. Arbitrage free pricing (cf. o10)
6. Forward contracts, futures, options (cf. o10)

There will also be more on life insurance.

Appendix A

Notation and introduction to probability

We cannot give a full development of required probability theory here, but we shall discuss some of the main concepts by introducing our notation.

In the preceding example, the redemption payment R is to be modelled as a random variable that can take the values 100 and 0. The important information about R is its distribution, that is given by probabilities $p_R(0) + p_R(100) = 1$. This specification allows to derive the distribution of related random variables like $S = (1+i)^{-n}R$.

Definition 34 The distribution of a *discrete* random variable R is represented by its set of possible outcomes $\{r_j : j \in \mathbb{N}\} \subset \mathbb{R}$ (e.g. $r_j = j$) and its *probability mass function* (p.m.f.) $p_R : \mathbb{R} \to [0,1]$ satisfying $p_R(r) = 0$ for all $r \notin \{r_j : j \in \mathbb{N}\}$ and $\sum_{j\in\mathbb{N}} p_R(r_j) = 1$. We write for $A \subset \mathbb{R}$

$$P(R \in A) = \sum_{j\in\mathbb{N}:r_j\in A} p_R(r_j).$$

To define a *multivariate* discrete random variable $(R_1, \ldots, R_n)$, we replace $\mathbb{R}$ by $\mathbb{R}^n$ in the above phrases and denote the p.m.f. by $p_{(R_1,\ldots,R_n)}$.

Finally, we define for functions $g : \{r_j : j \in \mathbb{N}\} \to \mathbb{R}$

$$E(g(R)) = \sum_{j \in \mathbb{N}} g(r_j) p_R(r_j)$$

provided the series converges.

Definition 35 The distribution of a *continuous* random variable R is represented by its Riemann integrable *probability density function* (p.d.f.) $f_R : \mathbb{R} \to [0, \infty)$ satisfying $\int_{-\infty}^{\infty} f_R(r)dr = 1$. We write for $A \subset \mathbb{R}$

$$P(R \in A) = \int_A f_R(r)dr = \int_{-\infty}^{\infty} 1_{\{r \in A\}} f_R(r)dr.$$

and for functions $g : \mathbb{R}^n \to \mathbb{R}$

$$E(g(R)) = \int_{-\infty}^{\infty} g(r) f_R(r)dr$$

provided the Riemann integrals exist.

To define *multivariate* continuous random variables $(R_1, \ldots, R_n)$, replace $\mathbb{R}$ by $\mathbb{R}^n$ and integrals by multiple integrals, denote the p.d.f. by $f_{(R_1,\ldots,R_n)}$.

Definition 36 For a (discrete *or* continuous) random variable R we define the *distribution function* F_R and the *survival function* $\bar{F}_R$ by

$$\begin{aligned} F_R(t) &= P(R \leq t) = P(R \in (-\infty, t]) \\ \bar{F}_R(t) &= P(R > t) = P(R \in (t, \infty)) = 1 - F_R(t), \qquad t \in \mathbb{R}. \end{aligned}$$

If they exist, we define the *mean* μ_R and the *variance* σ_R^2 of R as

$$\mu_R = E(R) \qquad \text{and} \qquad \sigma_R^2 = Var(R) = E((R - E(R))^2).$$

$\sigma_R = \sqrt{\sigma_R^2}$ is called the *standard deviation* of R.

Definition 37 (Discrete *or* continuous) Random variables $R_1, \ldots, R_n$ are said to be *independent* if

$$P((R_1, \ldots, R_n) \in A_1 \times \ldots \times A_n) = P(R_1 \in A_1) \ldots P(R_n \in A_n) \tag{1}$$

for all $A_1, \ldots, A_n \subset \mathbb{R}$ (such that the integrals exist in the continuous case). Here $P(R_j \in A_j) = P((R_1, \ldots, R_j, \ldots, R_n) \in \mathbb{R} \times \ldots \times A_j \times \ldots \times \mathbb{R})$ and we call the p.m.f. p_{R_j} or p.d.f. f_{R_j} of R_j the *marginal* p.m.f. or p.d.f.

Proposition 33 *R and S are independent if and only if (1) holds for all $A_j = (-\infty, t_j]$, $t_j \in \mathbb{R}$, $j = 1, \ldots, n$, if and only if*

$$p_{(R,S)}(r, s) = p_R(r) p_S(s) \qquad \text{or} \qquad f_{(R,S)}(r, s) = f_R(r) f_S(s)$$

for all $r, s \in \mathbb{R}$ in the respective discrete and continuous cases (for the continuous case we actually need some continuity assumptions or exceptional sets, that are irrelevant in practice. In fact p.d.f.'s are only essentially unique).

In practice, the dependence structure can always be derived from a small set of independent random variables, and dependencies only arise when these are transformed.

Example 67 Assume, that the life time T of a light bulb has a geometric distribution with parameter $q \in (0,1)$, i.e. $P(T = n) = (1-q)q^n$, $n = 0, 1, \ldots$. Then the random variables $B_n = 1_{\{T=n\}}$, i.e. $B_n = 1$ if $T = n$ and $B_n = 0$ if $T \neq n$, are Bernoulli variables with parameter $(1-q)q^n$. Of course, the B_n are not independent, since e.g.

$$P(B_0 = 1, B_1 = 1) = P(T = 0, T = 1) = 0 \neq P(B_0 = 1)P(B_1 = 1).$$

We recall that μ_R is the average value of R, also in the sense that

Proposition 34 (Weak law of large numbers, Tchebychev's inequality) *Given a sequence of independent, identically distributed random variables $(R_j)_{j\geq 1}$ with existing mean $\mu = \mu_{R_j}$, we have*

$$P\left(\left|\frac{1}{n}\sum_{j=1}^{n} R_j - \mu\right| > \varepsilon\right) \to 0 \qquad \textit{as } n \to \infty$$

for all $\varepsilon > 0$, i.e. $P - \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} R_j = \mu$, where $P - \lim$ denotes the limit in probability.

More precisely, if the variance $\sigma^2 = \sigma^2_{R_j}$ exists, $P\left(\left|\frac{1}{n}\sum_{j=1}^{n} R_j - \mu\right| > \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2}$.

σ^2_R is a measure for the spread of the distribution of R. Its definition $\sigma^2_R = E((R - \mu_R)^2)$ can be read as the expected (squared) deviation from the mean. Also the preceding proposition indicates that a high σ^2_R means more deviation from the mean.

Appendix B

A 1967-70 Mortality table

The following table is an extract from mortality tables of assured lives based on data collected between 1967 and 1970. These tables were used for examination purposes by the Faculty and Institute of Actuaries and are provided on their website at

http://www.actuaries.org.uk/students/FT-A1967-70.pdf

The extract below can be found at the top of page 7 of the pdf file.

Age $[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	Age $x+2$
53	.00376288	.00519413	.00844128	55
54	.00410654	.00570271	.00941902	56
55	.00447362	.00625190	.01049742	57
56	.00486517	.00684424	.01168566	58
57	.00528231	.00748245	.01299373	59
58	.00572620	.00816938	.01443246	60
59	.00619802	.00890805	.01601356	61
60	.00669904	.00970168	.01774972	62
61	.00723057	.01055365	.01965464	63
62	.00779397	.01146756	.02174310	64