1. Consider the DAG $\mathcal{D}$ with arrows $A \to C, B \to C, B \to D, C \to E, D \to F, E \to G, E \to H, F \to G, G \to J, I \to J$.

   (a) Find the moral graph $\mathcal{D}^m$ of $\mathcal{D}$;

   The moral graph is found by adding edges $AB, EF, GI$ and subsequently dropping directions.

   (b) Find a *minimal chordal cover* $\mathcal{G}$ of $\mathcal{D}^m$, i.e. a chordal graph $\mathcal{G} \supset \mathcal{D}^m$ with the property that removal of any edge in $\mathcal{G}$ which is not an edge in $\mathcal{D}^m$ will not be chordal;

   The moral graph $\mathcal{D}^m$ is not chordal as there is a chordless cycle $B \sim C \sim E \sim F \sim D \sim B$.

   A minimal chordal cover is obtained by adding edges $CD, DE$, or adding $CD, CF$, or adding $BE, DE$.

   (c) Arrange the cliques of $\mathcal{G}$ in a junction tree;

   Using the latter of the three chordal covers above, the cliques of the graph are $ABC, BCE, BDE, DEF, EFG, EH, GIJ$. These can be arranged in a junction tree by letting

   $$ABC \sim BCE \sim BDE \sim DEF \sim EFG \sim GIJ$$

   and finally attaching $EH$ to either of $BCE, BDE, DEF, EFG$.

   (d) For a specification of all conditional distributions $p_{v \mid \mathrm{pa}(v)}, v \in V$, allocate appropriate potentials to the junction tree to prepare for probability propagation.

   A possible allocation is as follows

   | ABC | BCE | BDE | DEF | EFG | EH | GIJ |
   |---|---|---|---|---|---|---|
   | $p_A p_B p_{C \mid AB}$ | $p_{E \mid C}$ | $p_{D \mid B}$ | $p_{F \mid DE}$ | $p_{G \mid EF}$ | $p_{H \mid E}$ | $p_I p_{J \mid GI}$ |

2. Consider random variables $X_1, \ldots, X_6$ taking values in $\{-1, 1\}$ and having distribution $P$ with joint probability mass function determined as

   $$p(x) \propto \exp\{\theta(x_1x_2 + x_2x_3 + x_2x_5 + x_3x_4 + x_3x_5 + x_3x_6)\},$$

   where $\theta \neq 0$.

   (a) Find the dependence graph of $P$ and identify its cliques;

   The dependence graph has cliques 12, 235, 36, 34;

   (b) Set up an appropriate junction tree for probability propagation;

   Attach 12 to 235, and then any two out of three links between 235, 34, and 36 can be used.

   (c) Allocate potentials to cliques;

   For all trees we allocate as follows:

   $$\begin{aligned} \phi_{12}(x_1, x_2) &= e^{\theta x_1 x_2} \\ \phi_{235}(x_2, x_3, x_5) &= e^{\theta(x_2x_3 + x_2x_5 + x_3x_5)} \\ \phi_{34}(x_3, x_4) &= e^{\theta x_3 x_4} \\ \phi_{36}(x_3, x_6) &= e^{\theta x_3 x_6}. \end{aligned}$$

(d) Modify potentials to conform with having observed that $X_1 = 1$ and $X_4 = 1$;

Potentials are changed to

$$\begin{aligned}
\phi_{12}(x_1, x_2) &= e^{\theta x_2}\delta(x_1, 1) \\
\phi_{235}(x_2, x_3, x_5) &= e^{\theta(x_2 x_3 + x_2 x_5 + x_3 x_5)} \\
\phi_{34}(x_3, x_4) &= e^{\theta x_3}\delta(x_4, 1) \\
\phi_{36}(x_3, x_6) &= e^{\theta x_3 x_6}.
\end{aligned}$$

(e) Calculate $P(X_6 = 1 \mid X_1 = 1, X_4 = 1)$ by probability propagation.

Since we only need a single marginal probability, we collect evidence to 36 as root.

We first send message from 12 to 235 by changing $\phi_{235}$ as

$$\phi_{235}(x_2, x_3, x_5) \leftarrow e^{\theta(x_2 x_3 + x_2 x_5 + x_3 x_5)} e^{\theta x_2} = e^{\theta(x_2 x_3 + x_2 x_5 + x_3 x_5 + x_2)}.$$

Next step depends on what tree we have chosen. If 36 is linked to 235 and 34, we next send message from 34 to 36 by letting

$$\phi_{36}(x_3, x_6) \leftarrow e^{\theta(x_3 x_6 + x_3)}.$$

Finally we have to send message from 235 to 36 as follows

$$\phi_{36}(x_3, x_6) \leftarrow e^{\theta(x_3 x_6 + x_3)} \sum_{x_2, x_5} \phi_{235}(x_2, x_3, x_5)$$

which yields

$$\phi_{36}(x_3, x_6) = e^{\theta(x_3 x_6 + x_3)} \left\{ e^{2\theta(x_3+1)} + 1 + e^{-2\theta} + e^{-2\theta x_3} \right\}.$$

Now $p(x_3, x_6 \mid X_1 = X_4 = 1) \propto \phi_{36}(x_3, x_6)$ as above, so we need to further marginalise over $x_3$ and renormalise, giving first

$$\begin{aligned}
&p(x_6 \mid X_1 = X_4 = 1) \propto \\
&\quad e^{\theta(x_6+1)} \left\{ e^{4\theta} + 1 + 2e^{-2\theta} \right\} + e^{-\theta(x_6+1)} \left\{ 1 + 1 + e^{-2\theta} + e^{2\theta} \right\}
\end{aligned}$$

and thus

$$\begin{aligned}
&P(X_6 = 1 \mid X_1 = X_4 = 1)/P(X_6 = -1 \mid X_1 = X_4 = 1) = \\
&\frac{e^{4\theta}\left\{e^{4\theta} + 1 + 2e^{-2\theta}\right\} + e^{-2\theta}\left\{(e^{-\theta} + e^{\theta})^2\right\}}{e^{4\theta} + 3 + 2e^{-2\theta} + e^{2\theta}}.
\end{aligned}$$

3. Consider a Gaussian distribution $\mathcal{N}_4(0, \Sigma)$ with

$$K = \Sigma^{-1} = \begin{pmatrix} 5 & 1 & 4 & 4 \\ 1 & 12 & 2 & 5 \\ 4 & 2 & 14 & 2 \\ 4 & 5 & 2 & 8 \end{pmatrix}.$$

(a) What is the marginal distribution of $X_1$?

The variance of $X_1$ is given by the upper left corner of the inverse of the concentration matrix, which is the ratio of two determinants:

$$\frac{954}{1926} = \frac{53}{107}$$

so $X_1 \sim \mathcal{N}(0, 53/107)$.

(b) Find the conditional distribution of $(X_2, X_3)$ given $(X_1 = 0, X_4 = 1)$;

The conditional concentration matrix is

$$K_{23\,|\,14} = \begin{pmatrix} 12 & 2 \\ 2 & 14 \end{pmatrix}$$

yielding the conditional covariance as its inverse

$$\Sigma_{23\,|\,14} = \frac{1}{164}\begin{pmatrix} 14 & -2 \\ -2 & 12 \end{pmatrix} = \frac{1}{82}\begin{pmatrix} 7 & -1 \\ -1 & 6 \end{pmatrix}.$$

To find the conditional expectation we first rearrange so that the order of variables is $2, 3, 1, 4$ yielding

$$\begin{pmatrix} 12 & 2 & 1 & 5 \\ 2 & 14 & 4 & 2 \\ 1 & 4 & 5 & 4 \\ 5 & 2 & 8 & 4 \end{pmatrix}.$$

The conditional expectation is thus

$$\begin{aligned} \mathbb{E}(X_2, X_3 \,|\, X_1 = 0, X_4 = 1)^\top &= -\frac{1}{82}\begin{pmatrix} 7 & -1 \\ -1 & 6 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= -\frac{1}{82}\begin{pmatrix} 7 & -1 \\ -1 & 6 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = -\frac{1}{82}\begin{pmatrix} 33 \\ 7 \end{pmatrix}. \end{aligned}$$

(c) Find the conditional distribution of $X_4$ given $X_2 = 0$.

We first find the conditional concentration matrix for $X_1, X_3, X_4$ by cutting out the appropriate bit:

$$K_{134\,|\,2} = \begin{pmatrix} 5 & 4 & 4 \\ 4 & 14 & 2 \\ 4 & 2 & 8 \end{pmatrix}.$$

The conditional variance of $X_4$ given $X_2$ is thus the lower right corner of the inverse of this matrix, which is the ratio of two determinants, so the conditional variance of $X_4$ is

$$\frac{54}{252} = \frac{3}{14}.$$

Thus $X_2 \,|\, X_4 = 0 \sim \mathcal{N}(0, 3/14)$.