1. For two discrete variables I and J with joint distribution p_{ij}, the *odds-ratio* θ_{ii*jj} is defined as

$$\theta_{ii^*jj^*} = \frac{p_{i\,|\,j}/p_{i^*\,|\,j}}{p_{i\,|\,j^*}/p_{i^*\,|\,j^*}} = \frac{p_{ij}p_{i^*j*}}{p_{i^*j}p_{ij*}}.$$

Show that $I \perp\!\!\!\perp J \iff \theta_{ii^*jj^*} \equiv 1$.

Strictly speaking, this is only true if $p_{ij} > 0$ as otherwise θ may not be well-defined and some care needs to be taken (defining $0/0 = 1$ for example).

If $I \perp\!\!\!\perp J$ we have $p(i\,|\,j) = p(i\,|\,j^*)$ and the result follows. If $\theta_{ii^*jj^*} \equiv 1$ and we choose a fixed i^*j^* with $p_{i^*j^*} > 0$, we have

$$p_{ij} = p_{i^*j}p_{ij*}/p_{i^*j^*}$$

and we have factorized p.

2. For three discrete variables I, J and K with joint distribution p_{ijk} the *conditional odds-ratio* $\theta_{ij\,|\,k}$ is similarly

$$\theta_{ii^*jj^*\,|\,k} = \frac{p_{i\,|\,jk}/p_{i^*\,|\,jk}}{p_{i\,|\,j^*k}/p_{i^*\,|\,j^*k}} = \frac{p_{ijk}p_{i^*j*k}}{p_{i^*jk}p_{ij*k}}.$$

Show that p belongs to the log-linear model with generating class $\{I, J\}, \{J, K\}\{I, K\}$ if and only if the conditional odds-ratio is constant in k. For simplicity, you may assume $p_{ijk} > 0$ for all i, j, k.

If $p_{ijk} = a_{ij}b_{jk}c_{ik}$ we have

$$\theta_{ii^*jj^*\,|\,k} = \frac{a_{ij}b_{jk}c_{ik}a_{i^*j^*}b_{j^*k}c_{i^*k}}{a_{i^*j}b_{jk}c_{i^*k}a_{ij^*}b_{j^*k}c_{ik}} = \frac{a_{ij}a_{i^*j^*}}{a_{i^*j}a_{ij^*}}.$$

Conversely, if the conditional odds-ratio is independent of k we can fix i^*, j^*, and some k^* and write

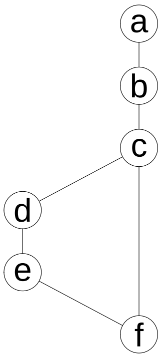
$$p_{ijk} = \theta_{ii^*jj^*\,|\,k^*}(p_{i^*jk}/p_{i^*j^*k})p_{ij^*k} = \tilde{a}_{ij}\tilde{b}_{jk}\tilde{c}_{ik}$$

where

$$\tilde{a}_{ij} = \theta_{ii^*jj^*\,|\,k^*}, \quad \tilde{b}_{jk} = p_{i^*jk}/p_{i^*j^*k}, \quad \tilde{c}_{ik} = p_{ij^*k}.$$

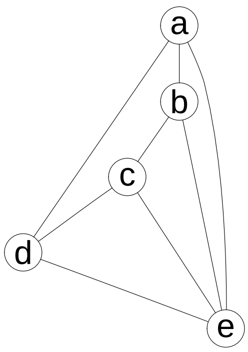
3. Draw the following graphs, identify their prime components, and verify whether they are chordal or not.

   (a) $E = \{\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{c,f\},\{e,f\}\}$.

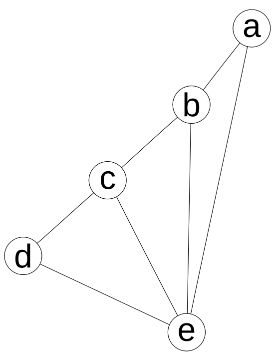
   The prime components are $\{a,b\}$, $\{b,c\}$, and $\{c,d,e,f\}$.
   This graph is not chordal as $\{c,d,e,f\}$ is not a clique.

   (b) $E = \{\{a,b\},\{b,c\},\{c,d\},\{d,a\},\{a,e\},\{b,e\},\{c,e\},\{d,e\}\}$.

   This graph is prime. It is not chordal because it is not a clique. Also, for example (a,b,c,d) form a 4-cycle without chord.

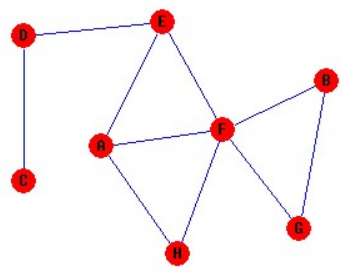
   (c) $E = \{\{a,b\},\{b,c\},\{c,d\},\{a,e\},\{b,e\},\{c,e\},\{d,e\}\}$.

   This graph is chordal. Its prime components are the cliques $\{a,b,e\}$, $\{b,c,e\}$, $\{c,d,e\}$.

4. Consider the generating class $\mathcal{A} = \{\{A, E, F\}, \{D, E\}, \{B, G, F\}, \{A, H, F\}, \{C, D\}\}$.

   (a) Find the dependence graph $G(\mathcal{A})$; is $\mathcal{A}$ conformal?
   The dependence graph is

   The generating class is conformal because it is exactly the cliques of the dependence graph.

   (b) Find a perfect numbering of the vertices with $F = 1$ using maximum cardinality search;
   An MCS-ordering could for example be:
   $$F, G, B, H, A, E, D, C.$$

   (c) Argue that $G(\mathcal{A})$ is chordal and $\mathcal{A}$ is decomposable;
   $G(\mathcal{A})$ is chordal because the above numbering is perfect. $\mathcal{A}$ is decomposable because $\mathcal{A}$ is conformal with a chordal dependence graph.

   (d) Arrange the cliques of the dependence graph $\mathcal{G}_{\mathcal{A}}$ in a junction tree and identify the separators and their multiplicities.
   For example
   $$BGF \sim AHF \sim AEF \sim ED \sim CD$$
   but also the tree with
   $$BGF \sim AEF \sim ED \sim CD, \quad AEF \sim AHF.$$
   The separators are the same for all valid junction trees, so in both cases: D, E, F, AF. All separators have multiplicity one.

   (e) Using the notation $n_{abcdefg}$ etc. for the cell counts of a contingency table corresponding to these variables, write an expression fof the MLE of the cell probabilities $p_{abcdefg}$.
   $$n\hat{p}_{abcdefg} = \frac{n_{aef}n_{ahf}n_{bgf}n_{cd}n_{de}}{n_d n_e n_f n_{af}}.$$

5. The regular multivariate Gaussian distribution over $\mathcal{R}^{|V|}$ has density

$$f(x \mid \xi, \Sigma) = (2\pi)^{-|V|/2} (\det K)^{1/2} e^{-(x-\xi)^\top K(x-\xi)/2},$$

where $K = \Sigma^{-1}$ is called the *concentration* of the distribution. If X is multivariate Gaussian, it holds that

$$\mathbb{E}(X) = \xi, \quad \mathrm{Cov}(X) = \Sigma$$

so ξ is the *mean* and Σ the *covariance* of X. We then write $X \sim \mathcal{N}_{|V|}(\xi, \Sigma)$.

Let X be multivariate normal $X \sim \mathcal{N}_{|V|}(0, \Sigma)$ and let $K = \Sigma^{-1}$. Show that the dependence graph $G(K)$ for X is given by

$$\alpha \not\sim \beta \iff k_{\alpha\beta} = 0.$$

Rewriting the Gaussian density we get

$$\log f(x) = \text{constant} - \frac{1}{2} \sum_{\alpha \in V} k_{\alpha\alpha} x_\alpha^2 - \sum_{\{\alpha,\beta\} \in E} k_{\alpha\beta} x_\alpha x_\beta.$$

which yields

$$\alpha \perp\!\!\!\perp \beta \mid V \setminus \{\alpha, \beta\} \iff k_{\alpha\beta} = 0.$$

6. Consider $X \sim \mathcal{N}_3(0, \Sigma)$. Show that $X_1 \perp\!\!\!\perp X_2 \mid X_3$ if and only if

$$\rho_{12} = \rho_{13}\rho_{23},$$

where ρ_{ij} denotes the correlation between X_i and X_j.

Without loss of generality we can assume $\sigma_{ii} = 1$ for all i so that

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}.$$

Using the cofactor formula we get that

$$k_{12} = \frac{-\det \begin{pmatrix} \rho_{12} & \rho_{23} \\ \rho_{13} & 1 \end{pmatrix}}{\det \Sigma} = \frac{\rho_{13}\rho_{23} - \rho_{12}}{\det \Sigma}.$$

Since $X_1 \perp\!\!\!\perp X_2 \mid X_3 \iff k_{12} = 0$, the result follows.