1. Prove that the following statements are all equivalent.

   (1) For all (x, y, z): $f(x, y, z)f(z) = f(x, z)f(y, z)$;

   (2) For all (x, y, z) with $f(z) > 0$: $f(x, y, z) = f(x \mid z)f(y, z)$;

   (3) For all (x, y, z) with $f(y, z) > 0$: $f(x \mid y, z) = f(x \mid z)$;

   (4) For all (x, y, z) with $f(y, z) > 0$: $f(x, z \mid y) = f(x \mid z)f(z \mid y)$;

   (5) For some functions h and k it holds: $f(x, y, z) = h(x, z)k(y, z)$.

   For simplicity we consider the case where $f(x, y, z) > 0$. The general case follows by taking care with zero entries.

   (1) $\Rightarrow$ (2):
   $$f(x, y, z) = \frac{f(x, z)f(y, z)}{f(z)} = f(x \mid z)f(y, z);$$

   (2) $\Rightarrow$ (3):
   $$f(x \mid y, z) = \frac{f(x, y, z)}{f(y, z)} = \frac{f(x \mid z)f(y, z)}{f(y, z)} = f(x \mid z);$$

   (3) $\Rightarrow$ (4):
   $$f(x, z \mid y) = f(x \mid y, z)f(z \mid y) = f(x \mid z)f(z \mid y);$$

   (4) $\Rightarrow$ (5):
   $$f(x, y, z) = f(x, z \mid y)f(y) = f(x \mid z)f(z \mid y)f(y) = h(x, z)k(y, z)$$
   where $h(x, z) = f(x \mid z)$ and $k(y, z) = f(z \mid y)f(y)$;

   (5) $\Rightarrow$ (1):
   Let $\bar{h}(z) = \sum_x h(x, z)$ and $\bar{k}(z) = \sum_y h(y, z)$. Then
   $$f(x, z) = \sum_y f(x, y, z) = \sum_y h(x, z)k(y, z) = h(x, z)\bar{k}(z)$$
   and
   $$f(y, z) = \sum_x f(x, y, z) = \sum_x h(x, z)k(y, z) = \bar{h}(z)k(y, z)$$
   and
   $$f(z) = \sum_{x,y} f(x, y, z) = \sum_{x,y} h(x, z)k(y, z) = \bar{h}(z)\bar{k}(z).$$
   Hence
   $$f(x, y, z)f(z) = h(x, z)k(y, z)\bar{h}(z)\bar{k}(z) = f(x, z)f(y, z).$$

2. Prove that for discrete random variables X, Y, Z, and W it holds that

   (C1) If $X \perp\!\!\!\perp Y \mid Z$ then $Y \perp\!\!\!\perp X \mid Z$;
   (C2) if $X \perp\!\!\!\perp Y \mid Z$ and $U = g(Y)$, then $X \perp\!\!\!\perp U \mid Z$;
   (C3) If $X \perp\!\!\!\perp Y \mid Z$ and $U = g(Y)$, then $X \perp\!\!\!\perp Y \mid (Z, U)$;
   (C4) If $X \perp\!\!\!\perp Y \mid Z$ and $X \perp\!\!\!\perp W \mid (Y, Z)$, then $X \perp\!\!\!\perp (Y, W) \mid Z$.

   (C1):
   This follows e.g. from (1) as this is symmetric in x and y;

   (C2):
   Let $f(x, y, z) = h(x, z)k(y, z)$. Then

$$f(x, u, z) = \sum_{y:g(y)=u} h(x, z)k(y, z) = h(x, z)\tilde{k}(u, z)$$

   where $\tilde{k}(u, z) = \sum_{y:g(y)=u} k(y, z)$.

   (C3):
   Let $f(x, y, z) = h(x, z)k(y, z)$. Then

$$f(x, y, u, z) = h(x, z)k(y, z)1_{\{g(y)=u\}}(y, u) = h(x, z)\check{k}(y, u, z),$$

   where

$$1_{\{g(y)=u\}}(y, u) = \begin{cases} 1 & \text{if } g(y) = u \\ 0 & \text{otherwise} \end{cases}$$

   and $\check{k}(u, z) = k(y, z)1_{\{g(y)=u\}}(y, u)$.

   (C4):
   Let $f(x, y, z) = h(x, z)k(y, z)$ from (5) and $f(w \mid x, y, z) = f(w \mid y, z)$ from (3). Then

$$f(x, y, z, w) = h(x, z)k(y, z)f(w \mid y, z) = h(x, z)\tilde{k}(y, z, w),$$

   with $\tilde{k}(y, z, w) = k(y, z)f(w \mid y, z)$. The conclusion follows from (5).

3. Show that for binary random variables (X, Y, Z) it holds that

$$X \perp\!\!\!\perp Y \text{ and } X \perp\!\!\!\perp Y \mid Z \Rightarrow (X, Z) \perp\!\!\!\perp Y \text{ or } X \perp\!\!\!\perp (Y, Z).$$

   Define the matrices A, B, and C as

$$a_{xy} = f(x, y); \quad b_{xz} = f(x \mid z); \quad c(z, y) = f(y, z).$$

   Since $X \perp\!\!\!\perp Y \mid Z$ we have

$$f(x, y) = \sum_z f(x, y, z) = \sum_z f(x, y, z) = \sum_z f(x \mid z)f(y, z) = \sum_z b_{xz}c_{zy}.$$

   In matrix formulation we can thus write

$$A = BC.$$

Since $X \perp\!\!\!\perp Y$ we have $f(x,y) = h(x)k(y)$ and all rows in A are proportional to the vector $k = \{k(y)\}$. Thus the matrix A has rank 1 and $\det A = 0$. But

$$\det(A) = \det(B)\det(C)$$

so we must either have $\det B = 0$ or $\det C = 0$. *In the binary case* the matrices are all 2×2 matrices so e.g.

$$\det B = 0 \iff \operatorname{rank} B = 1 \iff b_{xz} = u(x)v(z).$$

Hence in this case

$$f(x,y,z) = u(x)v(z)f(y,z) = \tilde{h}(x)\tilde{k}(y,z)$$

so $X \perp\!\!\!\perp (Y,Z)$.

In the case where $\det C = 0$ we similarly conclude $(X,Z) \perp\!\!\!\perp Y$.

In the non-binary case the situation is more complicated as we cannot deduce that either B or C has rank 1. In fact, we can only use *Sylvester's rank inequality* to conclude that

$$\operatorname{rank}(A) = 1 = \operatorname{rank}(BC) \geq \operatorname{rank}(B) + \operatorname{rank}(C) - |\mathcal{Z}|$$

where $|\mathcal{Z}|$ is the number of possible values of Z. For $|\mathcal{Z}| = 2$ we again obtain that either $\operatorname{rank}(B) = 1$ or $\operatorname{rank}(C) = 1$.

4. Shown that graph separation $\perp_{\mathcal{G}}$ in an undirected graph $\mathcal{G}$ is a compositional graphoid.

Symmetry any path from A to B is also a path from B to A and vice versa;

Decomposition If all paths from A to $B \cup D$ go via C this is obviously also true for all paths from A to B or A to D;

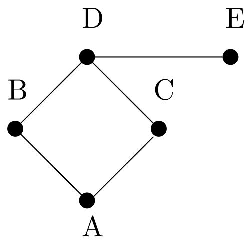
Weak union If all paths from A to B go via C, they clearly also go via $C \cup D$;

Contraction We must show that all paths from A to $B \cup D$ go via C. As this is assumed for paths to B we may focus on paths to D. But if such a path would go through B without intersecting C it would contradict the assumption that $A \perp_{\mathcal{G}} B \mid C$;

Intersection We must show that any path from A to B goes via D; if this is not the case, it must go through C as $A \perp_{\mathcal{G}} B \mid (C \cup D)$; but since $A \perp_{\mathcal{G}} C \mid B \cup D$, it cannot go through B before C, hence must go through D;

Composition This is bleedingly obvious...

5. Consider the graph below:



(a) Write down all conditional independence statements for this graph corresponding to the pairwise Markov property;

(b) Write down all conditional independence statements for this graph corresponding to the local Markov property;

(c) Write down some of the conditional independence statements for this graph which follow from the global Markov property and which are not listed above.

(a) $A \perp\!\!\!\perp D \mid B, C, E$;
$A \perp\!\!\!\perp E \mid B, C, D$;
$B \perp\!\!\!\perp C \mid A, D, E$;
$B \perp\!\!\!\perp E \mid A, C, D$;
$C \perp\!\!\!\perp E \mid A, B, D$;

(b) $A \perp\!\!\!\perp (D, E) \mid B, C$;
$B \perp\!\!\!\perp (C, E) \mid A, D$;
$C \perp\!\!\!\perp (B, E) \mid A, D$;
$D \perp\!\!\!\perp A \mid B, C, E$;
$E \perp\!\!\!\perp (A, B, C) \mid D$;

(c) For example:

$$A \perp\!\!\!\perp E \mid D, \quad A \perp\!\!\!\perp D \mid B, C, \quad B \perp\!\!\!\perp C \mid A, D, \quad E \perp\!\!\!\perp B \mid D.$$

6. *The result of this question is to be used in all remaining questions!* Show that if the distribution of $X \mid Z$ is degenerate so that X in effect is a deterministic function of Z, then $X \perp\!\!\!\perp Y \mid Z$ for all possible random variables Y.

This is a variant of the hopefully known fact that a constant random variable is independent of anything.

If $X \mid Z$ is degenerate we have

$$f(x \mid z) = \begin{cases} 1 & \text{if } x = g(z) \\ 0 & \text{otherwise} \end{cases}$$

and thus

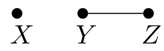
$$f(x, z) = \begin{cases} f(z) & \text{if } x = g(z) \\ 0 & \text{otherwise} \end{cases}$$

which, since $f(x,z) = \sum_y f(x,y,z)$, further implies

$$f(x,y,z) = \begin{cases} f(y,z) & \text{if } x = g(z) \\ 0 & \text{otherwise} \end{cases}$$

The factorization (1) of question 1 now gives the result.

7. Let $X = Y = Z$ with $P\{X = 1\} = P\{X = 0\} = 1/2$. Show that this distribution satisfies (P) but not (L) with respect to the graph below.

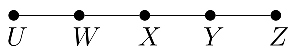


The pairwise Markov property says that $X \perp\!\!\!\perp Y \mid Z$ and $X \perp\!\!\!\perp Z \mid Y$, which both are satisfied, since X is a deterministic function of both Y and Z. However, we have that $\mathrm{bd}(X) = \emptyset$ so (L) would imply $X \perp\!\!\!\perp (Y, Z)$ which is false.

8. Let U and Z be independent with

$$P(U = 1) = P(Z = 1) = P(U = 0) = P(Z = 0) = 1/2,$$

$W = U$, $Y = Z$, and $X = WY$. Show that this distribution satisfies (L) but not (G) w.r.t. the graph below.



The local Markov property follows because all variables depend deterministically on their neighbours. But the global Markov property fails; for example it is false that $W \perp\!\!\!\perp Y \mid X$.