1. For two discrete variables I and J with joint distribution p_{ij}, the *odds-ratio* θ_{ii*jj} is defined as

$$\theta_{ii^*jj^*} = \frac{p_{i\,|\,j}/p_{i^*\,|\,j}}{p_{i\,|\,j^*}/p_{i^*\,|\,j^*}} = \frac{p_{ij}p_{i^*j*}}{p_{i^*j}p_{ij*}}.$$

Show that $I \perp\!\!\!\perp J \iff \theta_{ii^*jj^*} \equiv 1$.

2. For three discrete variables I, J and K with joint distribution p_{ijk} the *conditional odds-ratio* $\theta_{ij\,|\,k}$ is similarly

$$\theta_{ii^*jj^*\,|\,k} = \frac{p_{i\,|\,jk}/p_{i^*\,|\,jk}}{p_{i\,|\,j^*k}/p_{i^*\,|\,j^*k}} = \frac{p_{ijk}p_{i^*j*k}}{p_{i^*jk}p_{ij*k}}.$$

Show that p belongs to the log-linear model with generating class $\{I, J\}, \{J, K\}\{I, K\}$ if and only if the conditional odds-ratio is constant in k. For simplicity, you may assume $p_{ijk} > 0$ for all i, j, k.

3. Draw the following graphs, identify their prime components, and verify whether they are chordal or not:

   (a) $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{c, f\}, \{e, f\}\}$.

   (b) $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{a, e\}, \{b, e\}, \{c, e\}, \{d, e\}\}$.

   (c) $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, e\}, \{b, e\}, \{c, e\}, \{d, e\}\}$.

4. Consider the generating class $\mathcal{A} = \{\{A, E, F\}, \{D, E\}, \{B, G, F\}, \{A, H, F\}, \{C, D\}\}$.

   (a) Find the dependence graph $G(\mathcal{A})$; is $\mathcal{A}$ conformal?

   (b) Find a perfect numbering of the vertices with $F = 1$ using maximum cardinality search;

   (c) Argue that $G(\mathcal{A})$ is chordal and $\mathcal{A}$ is decomposable;

   (d) Arrange the cliques of the dependence graph $\mathcal{G}(\mathcal{A})$ in a junction tree and identify the separators and their multiplicities.

   (e) Using the notation $n_{abcdefg}$ etc. for the cell counts of a contingency table corresponding to these variables, write an expression fof the MLE of the cell probabilities $p_{abcdefg}$.

5. The regular multivariate Gaussian distribution over $\mathcal{R}^{|V|}$ has density

$$f(x\,|\,\xi, \Sigma) = (2\pi)^{-|V|/2}(\det K)^{1/2}e^{-(x-\xi)^\top K(x-\xi)/2},$$

where $K = \Sigma^{-1}$ is called the *concentration* of the distribution. If X is multivariate Gaussian, it holds that

$$\mathbb{E}(X) = \xi, \quad \mathrm{Cov}(X) = \Sigma$$

so ξ is the *mean* and Σ the *covariance* of X. We then write $X \sim \mathcal{N}_{|V|}(\xi, \Sigma)$. Let X be multivariate normal $X \sim \mathcal{N}_{|V|}(0, \Sigma)$ and let $K = \Sigma^{-1}$. Show that the dependence graph $G(K)$ for X is given by

$$\alpha \not\sim \beta \iff k_{\alpha\beta} = 0.$$

6. Consider $X \sim \mathcal{N}_3(0, \Sigma)$. Show that $X_1 \perp\!\!\!\perp X_2 \,|\, X_3$ if and only if

$$\rho_{12} = \rho_{13}\rho_{23},$$

where ρ_{ij} denotes the correlation between X_i and X_j.