1. Prove that the following statements are all equivalent, where f is a generic symbol for the density or probability mass function. For simplicity you may consider the discrete case only. In the general case "for all" should be read"except for a set of triplets (x, y, z) with probability zero".

   (1) For all (x, y, z): $f(x, y, z)f(z) = f(x, z)f(y, z)$;
   (2) For all (x, y, z) with $f(z) > 0$: $f(x, y, z) = f(x \mid z)f(y, z)$;
   (3) For all (x, y, z) with $f(y, z) > 0$: $f(x \mid y, z) = f(x \mid z)$;
   (4) For all (x, y, z) with $f(y, z) > 0$: $f(x, z \mid y) = f(x \mid z)f(z \mid y)$;
   (5) For some functions h and k it holds: $f(x, y, z) = h(x, z)k(y, z)$.

   Thus, any of these properties can be used to define the symbol $X \perp\!\!\!\perp Y \mid Z$.

2. Prove that for discrete random variables X, Y, Z, and W it holds that

   (C1) If $X \perp\!\!\!\perp Y \mid Z$ then $Y \perp\!\!\!\perp X \mid Z$;
   (C2) if $X \perp\!\!\!\perp Y \mid Z$ and $U = g(Y)$, then $X \perp\!\!\!\perp U \mid Z$;
   (C3) If $X \perp\!\!\!\perp Y \mid Z$ and $U = g(Y)$, then $X \perp\!\!\!\perp Y \mid (Z, U)$;
   (C4) If $X \perp\!\!\!\perp Y \mid Z$ and $X \perp\!\!\!\perp W \mid (Y, Z)$, then $X \perp\!\!\!\perp (Y, W) \mid Z$.

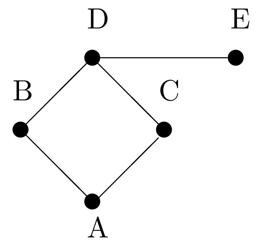
   Hint: Exploit the factorizations established in the first problem.

3. Show that for binary random variables (X, Y, Z) it holds that

$$X \perp\!\!\!\perp Y \text{ and } X \perp\!\!\!\perp Y \mid Z \Rightarrow (X, Z) \perp\!\!\!\perp Y \text{ or } X \perp\!\!\!\perp (Y, Z).$$

4. Shown that graph separation $\perp_{\mathcal{G}}$ in an undirected graph $\mathcal{G}$ is a compositional graphoid.

5. Consider the graph below:

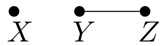


   (a) Write down all conditional independence statements for this graph corresponding to the pairwise Markov property;
   (b) Write down all conditional independence statements for this graph corresponding to the local Markov property;
   (c) Write down some of the conditional independence statements for this graph which follow from the global Markov property and which are not listed above.

Steffen L. Lauritzen, University of Oxford October 11, 2011

6. *The result of this question is to be used in all remaining questions!* Show that if the distribution of $X \mid Z$ is degenerate so that X in effect is a deterministic function of Z, then $X \perp\!\!\!\perp Y \mid Z$ for all possible random variables Y.

7. Let $X = Y = Z$ with $P\{X = 1\} = P\{X = 0\} = 1/2$. Show that this distribution satisfies (P) but not (L) with respect to the graph below.



8. Let U and Z be independent with

$$P(U = 1) = P(Z = 1) = P(U = 0) = P(Z = 0) = 1/2,$$

$W = U$, $Y = Z$, and $X = WY$. Show that this distribution satisfies (L) but not (G) w.r.t. the graph below.