1. Consider the causal Bayesian network with variables $A, B, C, D, E, F$ determined by $A, B, C$ being mutually independent and binary with values $\{-1, 1\}$ and $P(A = 1) = P(B = 1) = P(C = 1) = 1/2$ and $D = AB$, $E = BC$, $F = DE$.
   (a) Draw the graph of the associated Bayesian network
   (b) Calculate $P(C = 1 \mid A = -1)$ by probability propagation
   (c) Find the intervention probability $P(E = 1 \mid F \leftarrow -1, B \leftarrow 1)$.

2. Let $X_1, X_2, X_3, X_4, X_5$ be independent with $X_i \sim \mathcal{N}(0, 1)$. Define recursively the structural equation system

   $$Y_1 \leftarrow X_1, \quad Y_2 \leftarrow X_2{+}Y_1, \quad Y_3 \leftarrow X_3{+}Y_2, \quad Y_4 \leftarrow X_4{+}Y_2{+}Y_3, \quad Y_5 \leftarrow X_5{+}Y_1{+}Y_4$$

   and assume intervention in the system is made by replacement, so the associated Bayesian network is causal
   (a) Draw the causal DAG associated with this system
   (b) Find the concentration matrix $K = \Sigma^{-1}$ of $Y$.
   (c) Construct the dependence graph of $Y$;
   (d) Find the conditional distribution of $Y_5$ given $Y_3 = 0, Y_1 = 0$.
   (e) Find the intervention distribution of Find the intervention distribution of $Y_5$ given $Y_3 \leftarrow 0, Y_1 \leftarrow 0$.

3. Let $\mathcal{A} = \mathcal{C}$ be the cliques of a chordal graph $\mathcal{G} = (V, E)$. For $A \subseteq V$ let $H(A)$ denote the entropy of $X_A$.

   Show that
   $$H(V) = \sum_{C \in \mathcal{C}} H(C) - \sum_{S \in \mathcal{S}} \nu(S) H(S)$$
   where $\mathcal{S}$ are the minimal complete separators of $\mathcal{G}$ and $\nu(S)$ the number of times that the set $S$ appears as an intersection between neighbouring cliques in a junction tree for $\mathcal{A}$.

4. Consider a Gaussian distribution $\mathcal{N}_4(0, \Sigma)$ with $K = \Sigma^{-1}$ satisfying the conditional independence restrictions of the graph $\mathcal{G} = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}\{1, 4\}\{3, 4\}\}$.
   (a) Find two equations of degree 3 in $\sigma_{13}$ and $\sigma_{24}$ expressing these in terms of the variances $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{44}$ and the covariances $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}$; *Hint:* Express the appropriate inverse element of the covariance matrix as a cofactor;
   (b) Consider the likelihood equations based on observing a Wishart matrix $W = w$ with $W \sim \mathcal{W}(n, \Sigma)$. Use the answer under (a) to establish an equation of degree 5 for the maximum likelihood estimate of $\sigma_{13}$.
   (c) Assume next that $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{44} = 1$ and $\sigma_{12} = \sigma_{23} = \sigma_{34} = \rho$ and $\sigma_{14} = -\rho$. Show that then $\rho^2 < 1/2$.