1. Consider a Gaussian distribution $\mathcal{N}_5(0, \Sigma)$ with $K = \Sigma^{-1}$ satisfying the conditional independence restrictions of the graph $\mathcal{G} = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}\}$.

   (a) Show that the determinant of Σ satisfies

   $$\det \Sigma = \prod_{i=1}^{5} \sigma_{ii} \prod_{j=2}^{5} (1 - \rho_{1j}^2)$$

   where ρ_{ij} is the correlation between X_i and X_j;

   (b) Express the covariance σ_{23} in terms of the variances $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{44}, \sigma_{55}$ and the covariances $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}$.

2. Consider a Gaussian distribution $\mathcal{N}_4(0, \Sigma)$ with $K = \Sigma^{-1}$ satisfying the conditional independence restrictions of the graph $\mathcal{G} = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. Assume that the following Wishart matrix has been observed, with 10 degrees of freedom:

   $$\begin{pmatrix} 10 & 1 & 4 & 4 \\ 1 & 10 & 2 & 5 \\ 4 & 2 & 10 & 2 \\ 4 & 5 & 2 & 10 \end{pmatrix}.$$

   Find the maximum likelihood estimate of the concentration matrix.

3. Consider a directed graph $\mathcal{D} = (V, E)$ and assume given $k_v, v \in V$ with $k_v \geq 0$ and $\sum_{x_v \in \mathcal{X}_v} k_v(x_v \,|\, x_{\mathrm{pa}(v)}) = 1$. Define

   $$p(x) = \prod_{v \in V} k_v(x_v \,|\, x_{\mathrm{pa}(v)}).$$

   (a) Show that when $\mathcal{D}$ is acyclic, i.e. a DAG, this yields a well-defined probability distribution;

   (b) Show that it holds that

   $$p(x_v \,|\, x_{\mathrm{pa}(v)}) = k_v(x_v \,|\, x_{\mathrm{pa}(v)}). \tag{1}$$

   (c) Give a counterexample in the case where $\mathcal{D}$ has directed cycles.

   Hint: Use induction for (a) and (b), exploiting that a DAG always has a terminal vertex v_0, i.e. a vertex with no children.

4. Consider a DAG $\mathcal{D}$ with arrows $1 \to 2, 2 \to 5, 5 \to 6, 4 \to 5, 4 \to 7, 3 \to 4$.

   (a) List all conditional independence relations corresponding to the local, directed Markov property;

   (b) List all conditional independence relations corresponding to the ordered Markov property for the well-ordering induced by the given numbering;

(c) Find the ancestral sets generated by the following subsets:

i. $\{5\}$;

ii. $\{2, 7\}$;

iii. $\{4, 6\}$;

(d) Which of the following separation statements are true? For those that are not true, identify an active trail.

i. $2 \perp_{\mathcal{D}} 4 \mid 5$;

ii. $2 \perp_{\mathcal{D}} 7 \mid 3$,

iii. $1 \perp_{\mathcal{D}} 7 \mid 6$;

iv. $1 \perp_{\mathcal{D}} 3 \mid 4, 6$;

5. Let P be a distribution which factorizes over the DAG $\mathcal{D}$ and let $G(P)$ be its dependence graph. Show that $G(P) \subseteq \mathcal{D}^m$, where $\mathcal{D}^m$ is the moral graph of $\mathcal{D}$.