

1. Prove that the following statements are all equivalent, where f is a generic symbol for the density or probability mass function. For simplicity you may consider the discrete case only. In the general case “for all” should be read “except for a set of triplets (x, y, z) with probability zero”.

- (1) For all (x, y, z) : $f(x, y, z)f(z) = f(x, z)f(y, z)$;
- (2) For all (x, y, z) with $f(z) > 0$: $f(x, y, z) = f(x|z)f(y, z)$;
- (3) For all (x, y, z) with $f(y, z) > 0$: $f(x|y, z) = f(x|z)$;
- (4) For all (x, y, z) with $f(y, z) > 0$: $f(x, z|y) = f(x|z)f(z|y)$;
- (5) For some functions h and k it holds: $f(x, y, z) = h(x, z)k(y, z)$.

Thus, any of these properties can be used to define the symbol $X \perp\!\!\!\perp Y | Z$.

2. Prove that for discrete random variables X, Y, Z , and W it holds that

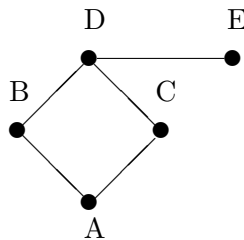
- (C1) If $X \perp\!\!\!\perp Y | Z$ then $Y \perp\!\!\!\perp X | Z$;
- (C2) if $X \perp\!\!\!\perp Y | Z$ and $U = g(Y)$, then $X \perp\!\!\!\perp U | Z$;
- (C3) If $X \perp\!\!\!\perp Y | Z$ and $U = g(Y)$, then $X \perp\!\!\!\perp Y | (Z, U)$;
- (C4) If $X \perp\!\!\!\perp Y | Z$ and $X \perp\!\!\!\perp W | (Y, Z)$, then $X \perp\!\!\!\perp (Y, W) | Z$.

Hint: Exploit the factorizations established in the first problem.

3. Show that for binary random variables (X, Y, Z) it holds that

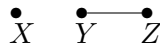
$$X \perp\!\!\!\perp Y \text{ and } X \perp\!\!\!\perp Y | Z \Rightarrow (X, Z) \perp\!\!\!\perp Y \text{ or } X \perp\!\!\!\perp (Y, Z).$$

4. Consider the graph below:



- (a) Write down all conditional independence statements for this graph corresponding to the pairwise Markov property;
- (b) Write down all conditional independence statements for this graph corresponding to the local Markov property;
- (c) Write down some of the conditional independence statements for this graph which follow from the global Markov property and which are not listed above.

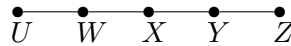
5. *The result of this question is to be used in all remaining questions!* Show that if the distribution of $X | Z$ is degenerate so that X in effect is a deterministic function of Z , then $X \perp\!\!\!\perp Y | Z$ for all possible random variables Y .
6. Let $X = Y = Z$ with $P\{X = 1\} = P\{X = 0\} = 1/2$. Show that this distribution satisfies (P) but not (L) with respect to the graph below.



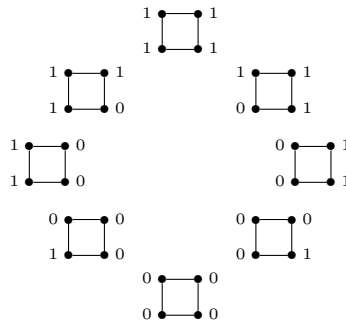
7. Let U and Z be independent with

$$P(U = 1) = P(Z = 1) = P(U = 0) = P(Z = 0) = 1/2,$$

$W = U$, $Y = Z$, and $X = WY$. Show that this distribution satisfies (L) but not (G) w.r.t. the graph below.



8. *This question might well be harder than the others!* Consider the distribution over four binary variables which gives probability $1/8$ to all of the 8 configurations displayed in the figure below:



Note that there are only four variables. The only reason that the graph (four-cycle) is repeated is to see the obvious pattern in the configuration.

Show that this distribution satisfies (G) with respect to the four cycle displayed, but the distribution does not factorize with respect to this graph, i.e., it does not satisfy (F).

Hint: Assume that it does factorize and show that if it is positive on these configurations, it must be positive on all 16 possible configurations of the four binary variables.