1. Consider $X \sim \mathcal{N}_3(0, \Sigma)$. Show that $X_1 \perp\!\!\!\perp X_2 \,|\, X_3$ if and only if
$$\rho_{12} = \rho_{13}\rho_{23},$$
where ρ_{ij} denotes the correlation between X_i and X_j.

2. Show that if $X \sim \mathcal{N}_3(0, \Sigma)$ with Σ regular, then $X_1 \perp\!\!\!\perp X_2$ and $X_1 \perp\!\!\!\perp X_2 \,|\, X_3$ implies that either $X_1 \perp\!\!\!\perp (X_2, X_3)$ or $(X_1, X_3) \perp\!\!\!\perp X_2$.

3. Consider a Gaussian distribution $\mathcal{N}_4(0, \Sigma)$ with
$$K = \Sigma^{-1} = \begin{pmatrix} 7 & 1 & 4 & 4 \\ 1 & 12 & 2 & 5 \\ 4 & 2 & 14 & 2 \\ 4 & 5 & 2 & 8 \end{pmatrix}.$$
What is the conditional distribution of X_2 given $(X_1 = 1, X_4 = 0)$?

4. Let X_1, X_2, X_3, X_4, X_5 be independent with $X_i \sim \mathcal{N}(0, 1)$. Define recursively
$$Y_1 \leftarrow X_1, \quad Y_2 \leftarrow X_2 + Y_1, \quad Y_3 \leftarrow X_3 + Y_2, \quad Y_4 \leftarrow X_4 + Y_3, \quad Y_5 \leftarrow X_5 + Y_4.$$
   (a) Find the covariance matrix Σ of Y;
   (b) Find the concentration matrix $K = \Sigma^{-1}$ of Y.
   (c) Construct the dependence graph of Y;
   (d) Find the conditional distribution of Y_3 given $Y_1 = y_1, Y_2 = y_2, Y_4 = y_4, Y_5 = y_5$.

5. Consider a Gaussian distribution $\mathcal{N}_5(0, \Sigma)$ with $K = \Sigma^{-1}$ satisfying the conditional independence restrictions of the graph $\mathcal{G} = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}\}$.
   (a) Show that the determinant of Σ satisfies
   $$\det \Sigma = \prod_{i=1}^{5} \sigma_{ii} \prod_{j=2}^{5} (1 - \rho_{1j}^2)$$
   where ρ_{ij} is the correlation between X_i and X_j;
   (b) Express the covariance σ_{23} in terms of the variances $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{44}, \sigma_{55}$ and the covariances $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}$.

6. Consider a Gaussian distribution $\mathcal{N}_4(0, \Sigma)$ with $K = \Sigma^{-1}$ satisfying the conditional independence restrictions of the graph $\mathcal{G} = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. Assume that the following Wishart matrix has been observed, with 10 degrees of freedom:
$$\begin{pmatrix} 10 & 1 & 4 & 4 \\ 1 & 10 & 2 & 5 \\ 4 & 2 & 10 & 2 \\ 4 & 5 & 2 & 10 \end{pmatrix}.$$
Find the maximum likelihood estimate of the concentration matrix.