1. (a) Define what is meant by saying that a distribution P *factorizes* w.r.t. a directed acyclic graph $\mathcal{D}$.

   (b) Define what is meant by saying that a distribution P satisfies the *local directed Markov property* w.r.t. a directed acyclic graph $\mathcal{D}$.

   (c) What is the relation between factorization and the local directed Markov property?

   (d) Consider a directed acyclic graph $\mathcal{D}$ with arrows $A \to B, B \to D, B \to E, C \to E, D \to E, D \to F, E \to F$.

      i. Form the moral graph $\mathcal{D}$.

      ii. Assume P satisfies the local directed Markov property with respect to $\mathcal{D}$. Which of the following statements can be concluded? Explain your reasoning?

$$C \perp\!\!\!\perp D \mid B, \quad A \perp\!\!\!\perp C \mid E, \quad B \perp\!\!\!\perp F \mid \{E, A\}.$$

   (e) Define what it means that a directed acyclic graph $\mathcal{D}'$ is Markov equivalent to $\mathcal{D}$?

   (f) What are the conditions for $\mathcal{D}$ and $\mathcal{D}'$ to be Markov equivalent?

   (g) Consider the following directed acyclic graphs obtained from $\mathcal{D}$ by reversing the arrows:

      i. $\mathcal{D}_1$ has reversed the arrow from $A \to B$, i.e. it has arrows $B \to A, B \to D, B \to E, C \to E, D \to E, D \to F, E \to F$;

      ii. $\mathcal{D}_2$ has reversed the arrow from $D \to E$, i.e. it has arrows $A \to B, B \to D, B \to E, C \to E, E \to D, D \to F, E \to F$.

   Which of these directed acyclic graphs are Markov equivalent to $\mathcal{D}$?

   The above question is from the Part C examination held in Trinity term 2008.

2. Consider the DAG $\mathcal{D}$ with arrows $A \to C, B \to C, B \to D, C \to E, D \to F, E \to G, E \to H, F \to G, G \to J, I \to J$.

   (a) Find the moral graph $\mathcal{D}^m$ of $\mathcal{D}$;

   (b) Find a *minimal chordal cover* $\mathcal{G}$ of $\mathcal{D}^m$, i.e. a chordal graph $\mathcal{G} \supset \mathcal{D}^m$ with the property that removal of any edge in $\mathcal{G}$ which is not an edge in $\mathcal{D}^m$ will not be chordal;

   (c) Arrange the cliques of $\mathcal{G}$ in a junction tree;

   (d) For a specification of all conditional distributions $p_{v \mid \mathrm{pa}(v)}, v \in V$, allocate appropriate potentials to the junction tree to prepare for probability propagation.

3. Consider random variables $X_1, \ldots, X_6$ taking values in $\{-1, 1\}$ and having distribution P with joint probability mass function determined as

$$p(x) \propto \exp\{\theta(x_1x_2 + x_2x_3 + x_3x_4 + x_3x_5)\},$$

where $\theta \neq 0$.

(a) Find the dependence graph of P and identify its cliques;

(b) Set up an appropriate junction tree for probability propagation;

(c) Allocate potentials to cliques;

(d) Calculate $P(X_5 = 1 \mid X_1 = 1, X_4 = 1)$ by probability propagation.