1. Consider a directed graph $\mathcal{D} = (V, E)$ and assume given $k_v, v \in V$ with $k_v \geq 0$ and $\sum_{x_v \in \mathcal{X}_v} k_v(x_v \,|\, x_{\mathrm{pa}(v)}) = 1$. Define
$$p(x) = \prod_{v \in V} k_v(x_v \,|\, x_{\mathrm{pa}(v)}).$$
   (a) Show that when $\mathcal{D}$ is acyclic, i.e. a DAG, this yields a well-defined probability distribution;
   (b) Show that it holds that
$$p(x_v \,|\, x_{\mathrm{pa}(v)}) = k_v(x_v \,|\, x_{\mathrm{pa}(v)});$$
   (c) Give a counterexample in the case where $\mathcal{D}$ has directed cycles.

   Hint: Use induction for (a) and (b), exploiting that a DAG always has a terminal vertex v_0, i.e. a vertex with no children.

2. Consider a DAG $\mathcal{D}$ with arrows $1 \to 2, 2 \to 5, 5 \to 6, 4 \to 5, 4 \to 7, 3 \to 4$.
   (a) List all conditional independence relations corresponding to the local, directed Markov property;
   (b) List all conditional independence relations corresponding to the ordered Markov property for the well-ordering induced by the given numbering;
   (c) Find the ancestral sets generated by the following subsets:
      i. $\{5\}$;
      ii. $\{2, 7\}$;
      iii. $\{4, 6\}$.
   (d) Which of the following separation statements are true? For those that are not true, identify an active trail.
      i. $2 \perp_{\mathcal{D}} 4 \,|\, 5$;
      ii. $2 \perp_{\mathcal{D}} 7 \,|\, 3$,
      iii. $1 \perp_{\mathcal{D}} 7 \,|\, 6$;
      iv. $1 \perp_{\mathcal{D}} 3 \,|\, 4, 6$.

3. Let P be a distribution which factorizes over the DAG $\mathcal{D}$ and let $G(P)$ be its dependence graph. Show that $G(P) \subseteq \mathcal{D}^m$, where $\mathcal{D}^m$ is the moral graph of $\mathcal{D}$.

4. A DAG $\mathcal{D}$ is said to be *perfect* if all parents are married, i.e. if it holds that
$$\alpha, \beta \in \mathrm{pa}(\gamma) \Rightarrow \alpha \to \beta \text{ or } \beta \to \alpha.$$
   (a) Show that a perfect DAG $\mathcal{D}$ is Markov equivalent to its *skeleton* i.e. the undirected graph obtained by ignoring directions on all arrows;
   (b) Show the converse, i.e. that if $\mathcal{D}$ is Markov equivalent to its skeleton, the $\mathcal{D}$ is perfect.
   (c) Show that the skeleton of a perfect $\mathcal{D}$ is chordal.
   (d) Show that the edges of a chordal graph $\mathcal{G}$ can be directed to create a Markov equivalent DAG.

   Hint: Exploit the existence of a perfect numbering of a chordal graph.