1. The regular multivariate Gaussian distribution over $\mathcal{R}^{|V|}$ has density

$$f(x \mid \xi, \Sigma) = (2\pi)^{-|V|/2}(\det K)^{1/2} e^{-(x-\xi)^\top K(x-\xi)/2},$$

where $K = \Sigma^{-1}$ is called the *concentration* of the distribution. If X is multivariate Gaussian, it holds that

$$\mathbf{E}(X) = \xi, \quad \mathrm{Cov}(X) = \Sigma$$

so ξ is the *mean* and Σ the *covariance* of X. We then write $X \sim \mathcal{N}_{|V|}(\xi, \Sigma)$.

Let X be multivariate normal $X \sim \mathcal{N}_{|V|}(0, \Sigma)$ and let $K = \Sigma^{-1}$. Show that the dependence graph $G(K)$ for X is given by

$$\alpha \not\sim \beta \iff k_{\alpha\beta} = 0.$$

2. Draw the following graphs, identify their prime components, and verify whether they are chordal or not:

   (a) $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{c, f\}, \{e, f\}\}$.

   (b) $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{a, e\}, \{b, e\}, \{c, e\}, \{d, e\}\}$.

   (c) $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, e\}, \{b, e\}, \{c, e\}, \{d, e\}\}$.

3. Consider the generating class $\mathcal{A} = \{\{A, E, F\}, \{D, E\}, \{B, G, F\}, \{A, H, F\}, \{C, D\}\}$.

   (a) Find the dependence graph $G(\mathcal{A})$; is $\mathcal{A}$ conformal?

   (b) Find a perfect numbering of the vertices with $F = 1$ using maximum cardinality search;

   (c) Argue that $G(\mathcal{A})$ is chordal and $\mathcal{A}$ is decomposable;

   (d) Arrange the cliques of the dependence graph $\mathcal{G}_\mathcal{A}$ in a junction tree and identify the separators and their multiplicities.

   (e) Using the notation $n_{abcdefg}$ etc. for the cell counts of a contingency table corresponding to these variables, write an expression fof the MLE of the cell probabilities $p_{abcdefg}$.

4. The *conditional entropy* $H(X \mid Y)$ is defined as the average entropy in the conditional distribution

$$H(X \mid Y) = \mathbf{E}[\mathbf{E}\{-\log f(X \mid Y) \mid Y\}] = \sum_y \left\{ \sum_x -f(x \mid y) \log f(x \mid y) \right\} f(y).$$

   (a) Use the information inequality to show that

$$H(X \mid Y) \leq H(X),$$

   i.e. the *entropy is always reduced by conditoning*;

   (b) Show that

$$H(X, Y) = H(X \mid Y) + H(Y).$$

(c) For three discrete random variables, show that

$$H(X,Y,Z) + H(Z) \leq H(X,Z) + H(Y,Z).$$

(d) Show further that

$$X \perp\!\!\!\perp Y \mid Z \iff H(X,Y,Z) + H(Z) = H(X,Z) + H(Y,Z).$$