1. Prove that the following statements are all equivalent, where f is a generic symbol for the density or probability mass function. For simplicity you may consider the discrete case only. In the general case "for all" should be read "except for a set of triplets (x, y, z) with probability zero".

   (1) For all (x, y, z): $f(x, y, z)f(z) = f(x, z)f(y, z)$;

   (2) For all (x, y, z) with $f(z) > 0$: $f(x, y, z) = f(x \mid z)f(y, z)$;

   (3) For all (x, y, z) with $f(y, z) > 0$: $f(x \mid y, z) = f(x \mid z)$;

   (4) For all (x, y, z) with $f(y, z) > 0$: $f(x, z \mid y) = f(x \mid z)f(z \mid y)$;

   (5) For some functions h and k it holds: $f(x, y, z) = h(x, z)k(y, z)$.

   Thus, any of these properties can be used to define the symbol $X \perp\!\!\!\perp Y \mid Z$.

2. Prove that for discrete random variables X, Y, Z, and W it holds that

   (C1) If $X \perp\!\!\!\perp Y \mid Z$ then $Y \perp\!\!\!\perp X \mid Z$;

   (C2) if $X \perp\!\!\!\perp Y \mid Z$ and $U = g(Y)$, then $X \perp\!\!\!\perp U \mid Z$;

   (C3) If $X \perp\!\!\!\perp Y \mid Z$ and $U = g(Y)$, then $X \perp\!\!\!\perp Y \mid (Z, U)$;

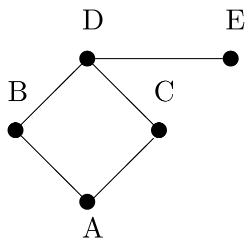
   (C4) If $X \perp\!\!\!\perp Y \mid Z$ and $X \perp\!\!\!\perp W \mid (Y, Z)$, then $X \perp\!\!\!\perp (Y, W) \mid Z$.

   Hint: Exploit the factorizations established in the first problem.

3. Show that for binary random variables (X, Y, Z) it holds that

$$X \perp\!\!\!\perp Y \text{ and } X \perp\!\!\!\perp Y \mid Z \Rightarrow (X, Z) \perp\!\!\!\perp Y \text{ or } X \perp\!\!\!\perp (Y, Z).$$

4. Consider the graph below:



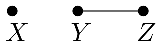
   (a) Write down all conditional independence statements for this graph corresponding to the pairwise Markov property;

   (b) Write down all conditional independence statements for this graph corresponding to the local Markov property;

   (c) Write down some of the conditional independence statements for this graph which follow from the global Markov property and which are not listed above.

5. *The result of this question is to be used in all remaining questions!* Show that if the distribution of $X \mid Z$ is degenerate so that X in effect is a deterministic function of Z, then $X \perp\!\!\!\perp Y \mid Z$ for all possible random variables Y.

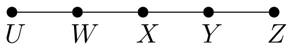
6. Let $X = Y = Z$ with $P\{X = 1\} = P\{X = 0\} = 1/2$. Show that this distribution satisfies (P) but not (L) with respect to the graph below.



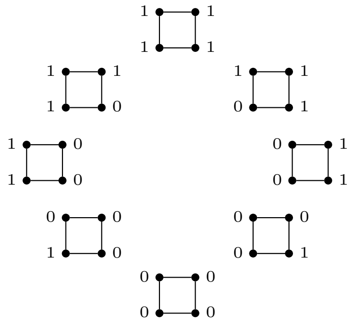
7. Let U and Z be independent with

$$P(U = 1) = P(Z = 1) = P(U = 0) = P(Z = 0) = 1/2,$$

$W = U$, $Y = Z$, and $X = WY$. Show that this distribution satisfies (L) but not (G) w.r.t. the graph below.



8. *This question might well be harder than the others!* Consider the uniform distribution on the 8 configurations displayed in the figure below:



Show that this distribution satisfies (G) but the distribution does not factorize, i.e., it does not satisfy (F).