Triangulated graphs and junction trees

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Decomposition of graphs

$\mathcal{G} = (V, E)$ undirected graph

(A, B, C) triple of disjoint subsets of V is a *decomposition* of $\mathcal{G}$, if and the following two conditions both hold:

1. $V = A \cup B \cup C$
2. C separates A from B;
3. C is a complete subset of V.

Decomposition is *proper* if $A \neq \emptyset$ and $B \neq \emptyset$.

A graph is *prime* if no proper decomposition exists.

Decomposable graphs

$\mathcal{G}$ is *decomposable* if either:

- $\mathcal{G}$ is complete, or
- There exists proper decomposition (A, B, C) such that both subgraphs $\mathcal{G}_{A \cup C}$ and $\mathcal{G}_{B \cup C}$ are decomposable.

Bad name! *Fully decomposable* much better.

Then 'decomposable' can be used to mean the opposite of prime.

Chordal and decomposable graphs

Graph is *chordal* if all cycles of length ≥ 4 have chords.

Other common names are *triangulated* graphs or *rigid circuit* graphs

Equivalent conditions for an undirected $\mathcal{G}$:

1. *$\mathcal{G}$ is decomposable,*
2. *$\mathcal{G}$ is chordal,*
3. *Every minimal (α, β)-separator is complete.*

Perfect numberings

A numbering $(v_1, \ldots, v_k)$ of the vertices V of an undirected graph $\mathcal{G}$ is called *perfect* if for all j, $\mathrm{bd}\, v_j \cap \{v_1, \ldots, v_{j-1}\}$, induce a complete subgraph.

An undirected graph is chordal if and only if it admits a perfect numbering.

If $\mathcal{G}$ is chordal and v is an arbitrary node of $\mathcal{G}$, then a perfect numbering of $\mathcal{G}$ exists with $v_1 = v$

Maximum cardinality search checks chordality of graph and constructs perfect numbering with complexity $O(|V| + |E|)$.

Maximum cardinality search

1. Choose arbitrary $v \in V$ and let $v_1 = v$;
2. When v_j has been chosen, choose v_{j+1} arbitrarily among the vertices with maximal number of numbered neighbours;
3. If at any stage $\mathrm{bd}(v_j) \cap \{v_1, \ldots, v_{j-1}\}$ is not complete, the graph is not chordal
4. Else, the graph is chordal and the final numbering $(v_1, \ldots, v_k)$ is perfect.

Note: Not all perfect numberings are MCS-generated!

Maximum cardinality search (formal)

- **Output**:= '$\mathcal{G}$ is chordal'.
- Counter $i := 1$, $L := \emptyset$, $c(v) := 0$ for all $v \in V$.
- While $L \neq V$:
  - $U := V \setminus L$.
  - Select any v maximizing $c(v)$ over $v \in U$, $v_i := v$.
  - If $\Pi(v_i) := \mathrm{bd}(v_i) \cap L$ is not complete in $\mathcal{G}$: **Output**:= '$\mathcal{G}$ is not chordal'. Otherwise, $c(w) := c(w) + 1$ for $w \in \mathrm{bd}(v_i) \cap U$.
  - $L := L \cup \{v_i\}$, $i := i + 1$.
- Report **Output**.

Finding the cliques of a chordal graph

Use numbering $(v_1, \ldots, v_k)$ obtained by maximum cardinality search, we can find the cliques of a chordal graph as follows. Let

$$\Pi(v_i) = \mathrm{bd}\, v_i \cap \{v_1, \ldots, v_{i-1}\}$$

and $\pi_i = |\Pi(v_i)|$

Call v_i a *ladder node* if $i = k$, or if $i < k$ and $\pi_{i+1} < 1 + \pi_i$. Let the jth ladder node, in ascending order, be λ_j, and define $C_j = \{\lambda_j\} \cup \Pi(\lambda_j)$.

There is a one-to-one correspondence between the ladder nodes and the cliques of $\mathcal{G}$, the clique associated with ladder node λ_j being C_j.

Running intersection property

The clique ordering $(C_1, C_2, \ldots)$ will possess the *running intersection property:*

$$C_j \cap (C_1 \cup \cdots \cup C_{j-1}) \subseteq C_i$$

for some $i < j$.

Junction tree can then easily be constructed, by linking C_j to any C_i satisfying RIP condition above.

Triangulation algorithms

If $\mathcal{G}$ is non-chordal, it is non-trivial problem to find a *optimal* triangulation (small cliques).

Any numbering $(v_1, \ldots, v_k)$ induces a triangulation by adding links, beginning with v_k, to complete $\mathrm{bd}(v_j) \cap \{v_1, \ldots, v_{j-1}\}$.

One-step-look-ahead algorithms finds first v_k so that $\mathrm{bd}(v_k)$ is 'close' to complete, fills-in and removes v_k, now using the same procedure to identify v_{k-1}, etc.

Algorithms exist which find optimal triangulations but in general they either return an optimal triangulation or give up at some point.