Graphical and Log-Linear Models

MSc Further Statistical Methods, Lecture 2
Hilary Term 2007

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Three-way tables

Admissions to Berkeley by department

Department	Sex	Whether admitted	
		Yes	No
I	Male	512	313
	Female	89	19
II	Male	353	207
	Female	17	8
III	Male	120	205
	Female	202	391
IV	Male	138	279
	Female	131	244
V	Male	53	138
	Female	94	299
VI	Male	22	351
	Female	24	317

Here are three variables A: Admitted?, S: Sex, and D: Department.

Conditional independence

For three variables it is of interest to see whether independence holds for fixed value of one of them, e.g. *is the admission independent of sex for every department separately?* We denote this as $A \perp\!\!\!\perp S \mid D$ and graphically as

Algebraically, this corresponds to the relations

$$p_{ijk} = p_{i+\,|\,k} p_{+j\,|\,k} p_{++k} = \frac{p_{i+k} p_{+jk}}{p_{++k}}.$$

Marginal and conditional independence

Note that there the two conditions

$$A \perp\!\!\!\perp S, \quad A \perp\!\!\!\perp S \mid D$$

are very different and will typically not both hold unless we either have $A \perp\!\!\!\perp (D, S)$ or $(A, D) \perp\!\!\!\perp S$, i.e. if one of the variables are completely independent of both of the others.

This fact is a simple form of what is known as *Yule–Simpson paradox.*

It can be much worse than this:

A *positive conditional association can turn into a negative marginal association* and vice-versa.

Admissions revisited

Admissions to Berkeley

Sex	Whether admitted	
	Yes	No
Male	1198	1493
Female	557	1278

Note this marginal table shows much lower admission rates for females.

Considering the departments separately, there is only a difference for department I, and it is the other way around...

Florida murderers

Sentences in 4863 murder cases in Florida over the six years 1973-78

	Sentence	
Murderer	Death	Other
Black	59	2547
White	72	2185

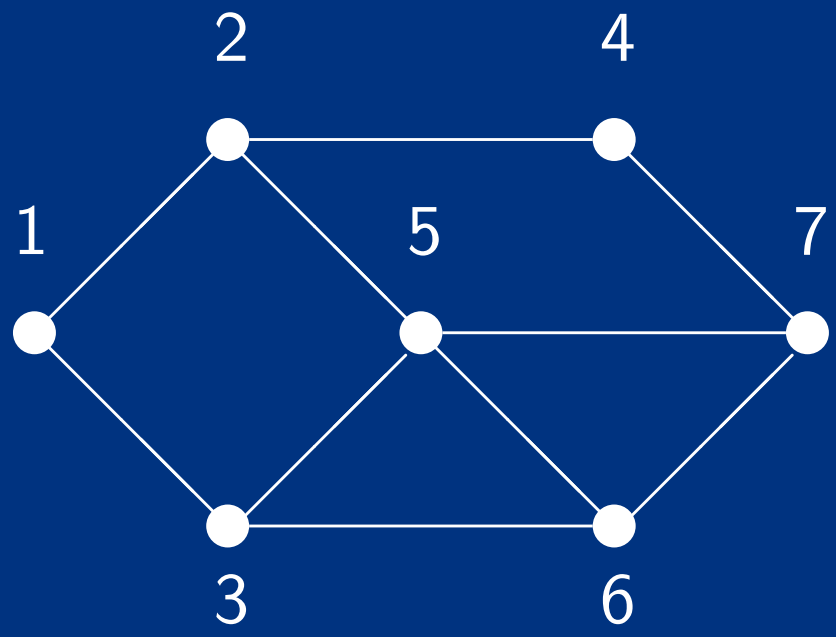
The table shows a greater proportion of white murderers receiving death sentence than black (3.2% vs. 2.3%), although the difference is not big, the picture seems clear.

Controlling for colour of victim

Victim	Murderer	Sentence	
		Death	Other
Black	Black	11	2309
	White	0	111
White	Black	48	238
	White	72	2074

Now the table for given colour of victim shows a very different picture. In particular, note that 111 white murderers killed black victims and none were sentenced to death.

Graphical models



For several variables, complex systems of conditional independence can be described by undirected graphs.

Then a set of variables A is conditionally independent of set B, given the values of a set of variables C if C *separates* A *from* B.

Conditional independence

Random variables X and Y are *conditionally independent* given the random variable Z if

$$\mathcal{L}(X \mid Y, Z) = \mathcal{L}(X \mid Z).$$

We then write $X \perp\!\!\!\perp Y \mid Z$

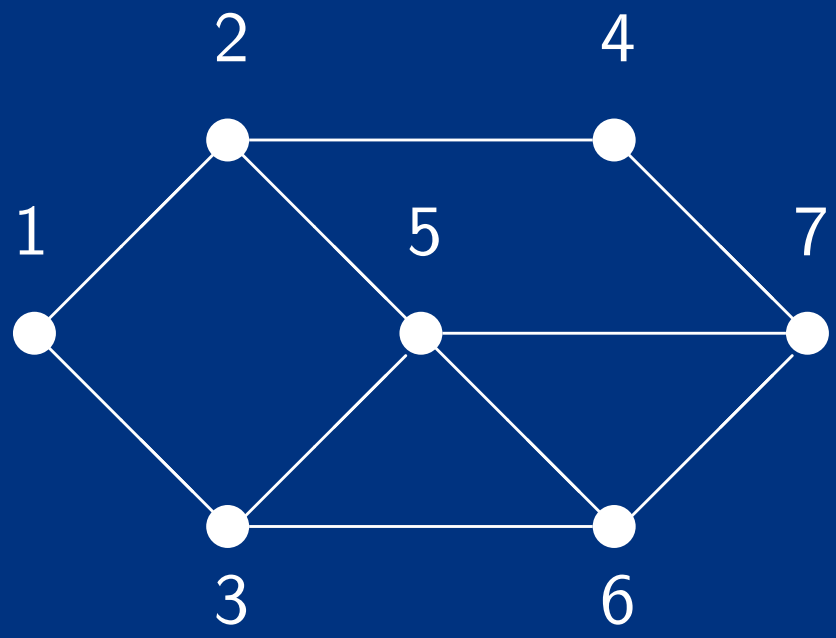
Intuitively:

Knowing Z renders Y *irrelevant* for predicting X.

Conditional independence can be expressed through Factorization of probabilities:

$$\begin{aligned} X \perp\!\!\!\perp Y \mid Z \quad &\iff \quad p_{xyz} p_{++z} = p_{x+z} p_{+yz} \\ &\iff \quad \exists a, b : p_{xyz} = a_{xz} b_{yz}. \end{aligned}$$

Graphical models



For several variables, complex systems of conditional independence can be described by undirected graphs.

A set of variables A is conditionally independent of set B, given the values of a set of variables C if *C separates A from B*.

Global Markov property and factorization

Formally we say for a given graph $\mathcal{G}$ that a distribution obeys *the global Markov property* (G) if

$$S \text{ separates } A \text{ from } B \text{ implies } A \perp\!\!\!\perp B \mid S.$$

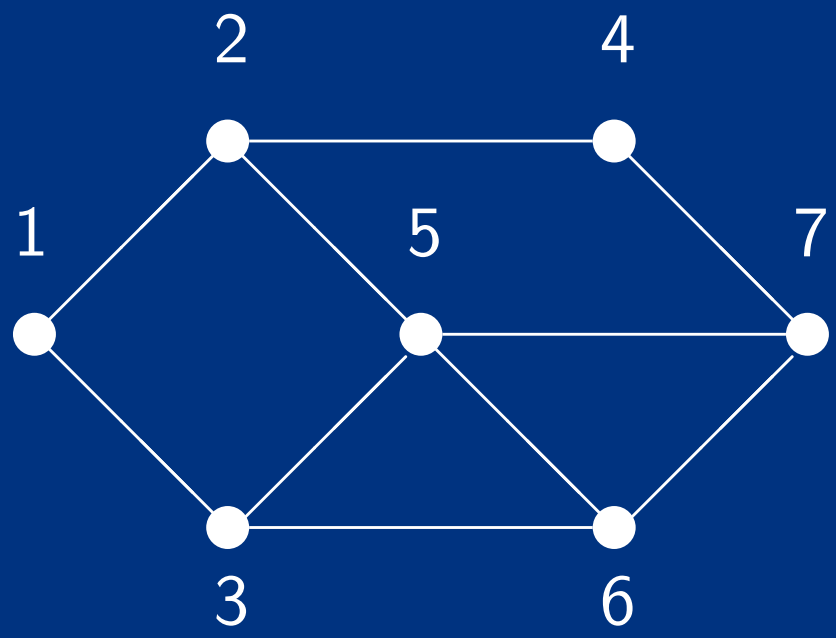
A distribution *factorizes* w.r.t. $\mathcal{G}$ if

$$p(x) = \prod_{a \text{ complete}} \psi_a(x)$$

where $\psi_a(x)$ depends on x through $x_a = (x_v)_{v \in a}$ only.

It can be shown that *a positive probability distribution is globally Markov w.r.t. a graph if and only if it factorizes as above.*

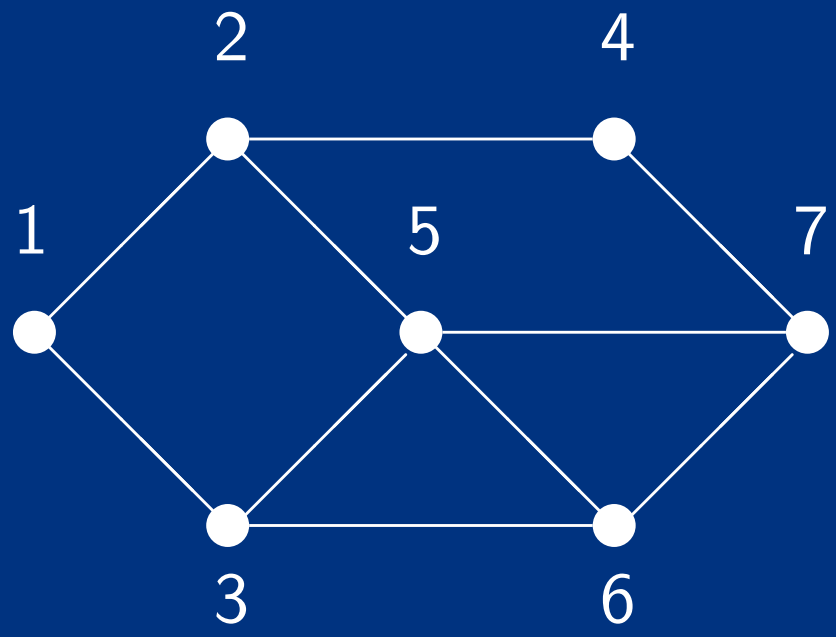
Global Markov property



To find conditional independence relations, one should look for separating sets, such as $\{2, 3\}$, $\{4, 5, 6\}$, or $\{2, 5, 6\}$

For example, it follows that $1 \perp\!\!\!\perp 7 \,|\, \{2, 5, 6\}$ and $2 \perp\!\!\!\perp 6 \,|\, \{3, 4, 5\}$.

Factorization



A probability distribution factorizes w.r.t. this graph iff it can be written in the form

$$\begin{aligned} p(x) &= \psi_{12}(x_1,x_2)\psi_{13}(x_1,x_3)\psi_{24}(x_2,x_4)\psi_{25}(x_2,x_5) \\ &\quad \times\psi_{47}(x_4,x_7)\psi_{356}(x_3,x_5,x_6)\psi_{567}(x_5,x_6,x_7) \end{aligned}$$

Log–linear models

$\mathcal{A} = \{a_1, \ldots, a_K\}$ denotes a set of (pairwise incomparable) subsets of $a_i \subseteq V$.

A probability distribution p (or function) *factorizes* w.r.t. $\mathcal{A}$ if it can be written as a product of terms where each only depend on variables in the same subset of $\mathcal{A}$, i.e. as

$$p(x) = \prod_{a \in \mathcal{A}} \psi_a(x)$$

where $\psi_a(x)$ depends on x through $x_a = (x_v)_{v \in a}$ only.

The set of distributions which factorize w.r.t. $\mathcal{A}$ is the *log–linear model* generated by $\mathcal{A}$.

$\mathcal{A}$ is the *generating class* of the log–linear model.

If the distribution factorizes without being everywhere positive, it will also satisfy all the Markov properties, but not the other way around.

Formally, we define *the graphical model with graph* $G = (V, E)$ to be the log-linear model with $\mathcal{A} = \mathcal{C}$, where $\mathcal{C}$ are the *cliques* (i.e. maximal complete subsets) of the graph.

Example

Consider a three way contingency table, where e.g. m_{ijk} denotes the mean of the counts N_{ijk} in the cell (i, j, k) which has then been expanded as e.g.

$$\log m_{ijk} = \alpha_i + \beta_j + \gamma_k \tag{1}$$

or

$$\log m_{ijk} = \alpha_{ij} + \beta_{jk} \tag{2}$$

or

$$\log m_{ijk} = \alpha_{ij} + \beta_{jk} + \gamma_{ik}, \tag{3}$$

or (with redundancy)

$$\log m_{ijk} = \gamma + \delta_i + \phi_j + \eta_k + \alpha_{ij} + \beta_{jk} + \gamma_{ik},$$

The additive terms in the expansion are known as *interaction terms of order* $|a| - 1$ or $|a|$*-factor interactions*.

Interaction terms of 0th order are called *main effects*.

Dependence graph of log–linear model

For any generating class $\mathcal{A}$ we can construct the dependence graph of the corresponding log–linear model.

This is determined by the relation

$$\alpha \sim \beta \iff \exists a \in \mathcal{A} : \alpha, \beta \in a.$$

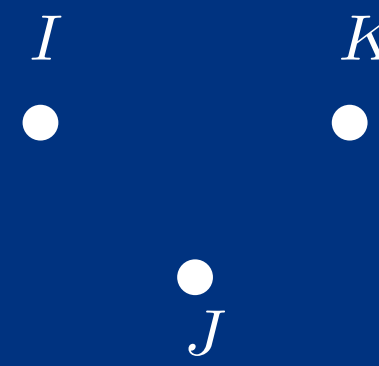
Then any probability distribution which factorizes w.r.t. $\mathcal{A}$ also satisfies the global Markov property w.r.t. $G(\mathcal{A})$.

This is by default the graph displayed in MIM.

Independence

The log–linear model specified by (1) is known as the *main effects model.*

It has generating class consisting of singletons only $\mathcal{A} = \{\{I\}, \{J\}, \{K\}\}$. It has dependence graph

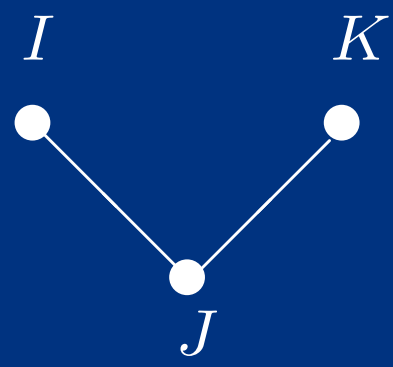


Thus it corresponds to *complete independence*.

Conditional independence

The log–linear model specified by (2) has no interaction between I and K.

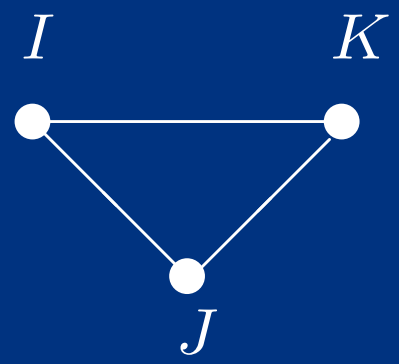
It has generating class $\mathcal{A} = \{\{I, J\}, \{J, K\}\}$ and dependence graph



Thus it corresponds to the *conditional independence* $I \perp\!\!\!\perp K \,|\, J$.

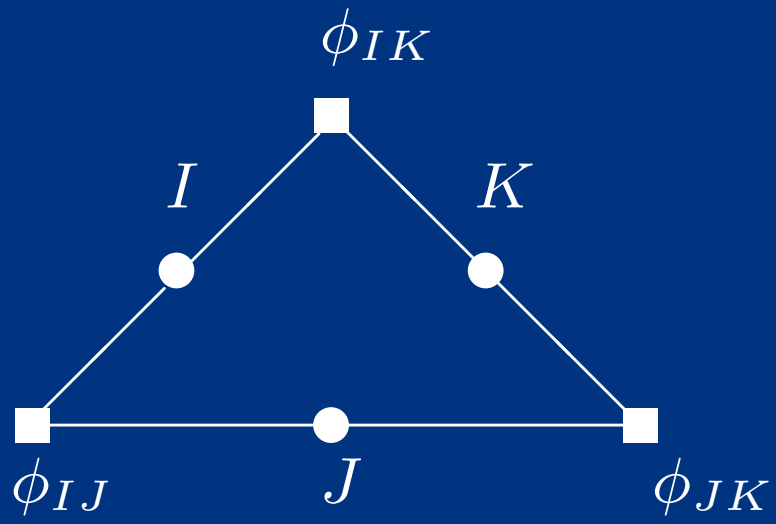
No interaction of second order

The log–linear model specified by (3) has no second-order interaction. It has generating class $\mathcal{A} = \{\{I, J\}, \{J, K\}, \{I, K\}\}$ and its dependence graph



is the complete graph. Thus it has no conditional independence interpretation.

Interaction graphs



The *interaction graph* of $\mathcal{A}$ is the graph with vertices $V \cup \mathcal{A}$ and edges define by

$$\alpha \sim a \iff \alpha \in a.$$

Using this graph all log–linear models admit a simple visual representation. Can be requested in MIM.

Likelihood function

The likelihood function for an unknown p can be expressed as

$$L(p) = \prod_{\nu=1}^{n} p(x^{\nu}) = \prod_{x \in \mathcal{X}} p(x)^{n(x)}.$$

In contingency table form the data follow a multinomial distribution

$$P\{N(x) = n(x), x \in \mathcal{X}\} = \frac{n!}{\prod_{x \in X} n(x)!} \prod_{x \in \mathcal{X}} p(x)^{n(x)}$$

but this only affects the likelihood function by a constant factor.

It can be shown that in log-linear models, *the likelihood function has at most one maximum*. When zero-values are allowed, it *always* has one.

MIM uses an algorithm for fitting known as *Iterative Proportional Fitting* which, if properly implemented, also works in the case where probabilities are allowed to be zero (sparse tables).

Also implemented e.g. in *R* in `loglin` with front end `loglm` in MASS.

An alternative is to "pretend" that counts are independent and Poisson distributed and use `glm`. However, the algorithm used there does *not* work when estimated cell probabilities are zero.