Conditional Independence and Graphical Models

MSc Further Statistical Methods, Lecture 2
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Three-way tables

Admissions to Berkeley by department

Department	Sex	Whether admitted	
		Yes	No
I	Male	512	313
	Female	89	19
II	Male	353	207
	Female	17	8
III	Male	120	205
	Female	202	391
IV	Male	138	279
	Female	131	244
V	Male	53	138
	Female	94	299
VI	Male	22	351
	Female	24	317

Here are three variables A: Admitted?, S: Sex, and D: Department.

Conditional independence

For three variables it is of interest to see whether independence holds for fixed value of one of them, e.g. *is the admission independent of sex for every department separately?*. We denote this as $A \perp\!\!\!\perp S \mid D$ and graphically as

Algebraically, this corresponds to the relations

$$p_{ijk} = p_{i+\,|\,k}\,p_{+j\,|\,k}\,p_{++k} = \frac{p_{i+k}\,p_{+jk}}{p_{++k}}.$$

Marginal and conditional independence

Note that there the two conditions

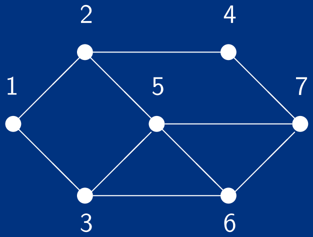
$$A \perp\!\!\!\perp S, \quad A \perp\!\!\!\perp S \mid D$$

are very different and will typically not both hold unless we either have $A \perp\!\!\!\perp (D, S)$ or $(A, S) \perp\!\!\!\perp D$, i.e. if one of the variables are completely independent of both others.

This fact is a simple form of what is known as *Yule–Simpson paradox.* It can be much worse than this:

A *positive conditional association can turn into a negative marginal association* and vice-versa.

Graphical models



For several variables, complex systems of conditional independence can be described by undirected graphs.

Then a set of variables A is conditionally independent of set B, given the values of a set of variables C if C *separates A from B*.

For example in picture above

$$1 \perp\!\!\!\perp \{4,7\} \,|\, \{2,3\}, \qquad \{1,2\} \perp\!\!\!\perp 7 \,|\, \{4,5,6\}.$$

Algebraically the picture represents the fact that the joint probability of all variables factorizes into terms that only depends on *cliques* of the graph. In the example:

$$p_{ijklmno} = a_{ij} b_{ik} c_{jm} c_{jl} c_{kmn} c_{lo} c_{mno}.$$