1. Let $X = (X_1, \dots, X_n)$ be a sample from the Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ both unknown, i.e. the distribution with individual densities
$$f(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x}, \quad x > 0.$$
   (a) Show that the asymptotic covariance matrix of $(\hat{\alpha}, \hat{\beta})$ is equal to
$$\frac{1}{n\{\alpha\psi'(\alpha) - 1\}} \begin{pmatrix} \alpha & \beta \\ \beta & \beta^2\psi'(\alpha) \end{pmatrix},$$
   where $\psi'(\alpha)$ is the Trigamma function.
   *Take care:* The original overheads from Lecture 6 had an incorrect sign in the Gamma example. The overheads which are on the web now should be correct on this point.
   (b) Show that $\hat{\alpha}$ and $\hat{\gamma}$ are asymptotically independent, where
$$\gamma = \mathbf{E}(X) = \alpha/\beta.$$
   *Hint:* Use the delta method and the result under (a) above.

2. Consider a sample $X = (X_1, \dots, X_n)$ from a normal distribution $\mathcal{N}(\mu, \mu^2)$, where $\mu > 0$ is unknown. This corresponds to the coefficient of variation $\sqrt{\mathbf{V}(X)}/\mathbf{E}(X)$ being known and equal to 1.
   (a) Find the score function for $\mu$;
   (b) Show that the likelihood equation has a unique root $\hat{\mu}$ within the parameter space unless $X_i$ are all equal to zero;
   (c) Calculate the observed information at $\hat{\mu}$ and use this to show that the root $\hat{\mu}$ is indeed the MLE of $\mu$;
   (d) Show that the asymptotic variance of $\hat{\mu}$ is equal to $\mu^2/(3n)$. You may without proof assume that the necessary regularity conditions are satisfied;
   (e) Consider the estimator
$$\tilde{\mu} = \left\{\bar{X} + 2\sqrt{SSD/(n-1)}\right\}/3,$$
   where $SSD$ is the *S*um of *S*quared *D*eviations
$$SSD = \sum_{i=1}^{n} (X_i - \bar{X})^2.$$
   Show that also $\tilde{\mu}$ is asymptotically unbiased and asymptotically efficient;
   *Hint:* Use the delta method on the function $g(x) = \sqrt{x}$.

3. Consider a sample $X = (X_1, \ldots, X_n)$ from a distribution with an individual density satisfying Cramér's conditions in the one-dimensional case and assume that the MLE $\hat{\theta}$ is a consistent root of the likelihood equation so that $\hat{\theta} \xrightarrow{P} \theta_0$ where $\theta_0$ is the true value of the parameter.

   You may now without proof use the fact that these conditions imply that *the Fisher information* $i(\theta)$ *is continuous at* $\theta_0$.

   (a) Show that
   $$\sqrt{n i(\hat{\theta})}(\hat{\theta} - \theta_0) \stackrel{\mathrm{a}}{\sim} \mathcal{N}(0, 1).$$
   *Hint:* Use Slutsky's theorem and the continuity of $i(\theta)$ at $\theta_0$;

   (b) Show that
   $$j_n(\hat{\theta})/n = \frac{-1}{n} \sum_{k=1}^{n} \frac{\partial^2}{\partial \theta^2} \log f(X_k; \hat{\theta}) \xrightarrow{P} i(\theta_0).$$
   *Hint:* Use Taylor's theorem and the boundedness of the third derivative.

   (c) Show that
   $$\sqrt{j_n(\hat{\theta})}(\hat{\theta} - \theta_0) \stackrel{\mathrm{a}}{\sim} \mathcal{N}(0, 1).$$
   *Hint:* Use Slutsky's theorem and the result under (b).