1. Let $X = (X_1, \dots, X_n)$ be a sample from an *exponential distribution* with individual densities
$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, x > 0,$$
where $\theta > 0$ is unknown.

   (a) Show that $\tilde{\theta}^2 = X_1^2/2$ is an unbiased estimator of $g(\theta) = \theta^2$;

   (b) Show that $t(X) = \sum X_i$ is sufficient for θ;

   (c) Rao–Blackwellize $\tilde{\theta}^2$ to find an improved unbiased estimator of $g(\theta)$;
   Hint: You may use without proof that the conditional distribution of $U = X_1/(\sum X_i)$ follows a Beta-distribution $\mathcal{B}(1, n-1)$, where $\mathcal{B}(\alpha, \beta)$ is the distribution with density
$$f(t;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1}, 0 < t < 1.$$

   (d) Is the Rao–Blackwellized estimator efficient?

   (e) Find the MLE of θ^2 and compare the mean square error of the two estimates.

2. Let $X = (X_1, \dots, X_n)$ be independent with all individual densities $f_j(x_j;\theta)$ for $X_j, j = 1, \dots, n$ satisfying usual conditions of regularity and smoothness. Let $i_j(\theta)$ denote the Fisher information about θ in X_j and let $i(\theta)$ be the Fisher information in X.

   (a) Show that the Fisher information is *additive* over independent experiments:
$$i(\theta) = i_1(\theta) + \cdots + i_n(\theta).$$

   (b) Next, let $Y = t(X)$ be a transformation of X and denote the Fisher information about θ by $i_Y(\theta)$. Similarly, let $i_{X\,|\,y}(\theta)$ denote the Fisher information conditionally on $Y = y$ and $i_{X\,|\,Y}(\theta)$ its expectation. Show, at least in the discrete case, that
$$i(\theta) = i_{X\,|\,Y}(\theta) + i_Y(\theta).$$

   (c) Using the result under (b), show that
$$i(\theta) \geq i_Y(\theta)$$
   with equality if and only if $Y = t(X)$ is sufficient for θ.

The quantity $i_{X \mid Y}(\theta)$ is known as the *loss of information* associated with the transformation $Y = t(X)$. A sufficient transformation does not lose information in this sense.

Hint: Use that the information satisfies

$$i(\theta) = \mathbf{E}_\theta \left\{ -\frac{\partial^2}{\partial\theta^2} \log f(X;\theta) \right\}$$

and factorize the density of X appropriately.

3. Let $X = (X_1, \ldots, X_n)$ be a sample of size n from the uniform distribution on the interval $(\mu - \delta, \mu + \delta)$

$$f(x;\mu,\delta) = (2\delta)^{-1} \text{ for } \mu - \delta < x < \mu + \delta \text{ and } 0 \text{ otherwise.}$$

and consider $\theta = (\delta, \mu)$ with $\delta > 0$ and μ both unknown.

(a) Show that $(X^{(1)}, X^{(n)})$ is minimal sufficient for θ;

(b) Estimate $\theta = (\delta, \mu)$ by the method of moments and denote the estimator by $\tilde{\theta} = (\tilde{\mu}, \tilde{\delta})$;

(c) Show that the maximum likelihood estimator $\hat{\theta}$ is given by

$$\hat{\mu} = (X_{(1)} + X_{(n)})/2, \quad \hat{\delta} = (X_{(1)} - X_{(n)})/2;$$

(d) Show that the MLE of μ is unbiased;

(e) Find the relative efficiency of $\tilde{\mu}$ to $\hat{\mu}$.

(f) Consider now the case where $\delta = 1$ is *known*. Show in this case that $A = (X_{(1)} - X_{(n)})/2$ is ancillary.

(g) Find the conditional density of the estimator $\hat{\mu} = (X_{(1)} + X_{(n)})/2$, given $A = a$ under the assumption $\delta = 1$.