1. Let $X = (X_1, \ldots, X_n)$ be independent and identically normally distributed as $\mathcal{N}(\mu, \sigma^2)$, where μ and σ^2 are both unknown, and assume $n = 2p + 1$ is odd.

   Define the estimators

   $$\hat{\mu} = \bar{X} = (X_1 + \cdots + X_n)/n, \quad \tilde{\mu} = \text{med}(X) = X_{(p+1)},$$

   and

   $$\hat{\sigma} = S = \sqrt{\sum_1^n (X_i - \bar{X})^2/(n-1)}.$$

   (a) Show that $\hat{\mu}$ is an unbiased estimator of μ;

   (b) Show that $\tilde{\mu}$ is an unbiased estimator of μ;
   Hint: You should exploit the symmetry of the distribution of $X_i - \mu$ and the identity

   $$X_{(p+1)} - \mu = (X - \mu)_{(p+1)} = -\{(\mu - X)_{(p+1)}\}.$$

   (c) Show that $\hat{\sigma}$ is not an unbiased estimator of σ;

   (d) Find an unbiased estimator of σ of the form $\check{\sigma} = cS$ in the case $p = 1$ $(n = 3)$.
   Hint: The Gamma function

   $$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

   has the properties

   $$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha) \text{ and } \Gamma(1/2) = \sqrt{\pi}.$$

   (e) Same as in (d), but for general n.

2. Let $X = (X_1, \ldots, X_n)$ be a sample of size n from the *uniform* distribution on the interval $(0, \theta)$ with density for a single observation being

   $$f(x; \theta) = 1/\theta \text{ for } 0 < x < \theta \text{ and } 0 \text{ otherwise}$$

   and consider $\theta > 0$ unknown.

   (a) Find the variance of the estimator $\tilde{\theta} = \bar{X}$;

   (b) Sketch the likelihood function;

   (c) Find the mean of the (maximum likelihood) estimator $\hat{\theta} = X_{(n)}$, i.e. the largest observation;

(d) Find an unbiased estimator of the form $\check{\theta} = cX_{(n)}$ and calculate its variance;

(e) Compare the mean square errors of $\hat{\theta}$, $\tilde{\theta}$, and $\check{\theta}$.

3. Consider $X_1, \ldots, X_n$ to be independent with means $\mathbf{E}(X_i) = \mu + \beta_i$ and variances $\mathbf{V}(X_i) = \sigma_i^2$. Such a situation could for example occur when X_i are estimators of μ obtained from independent sources and β_i is the *bias* of the estimator X_i.

   We now consider pooling the estimators of μ into a common estimator by using a linear combination:

$$\hat{\mu} = w_1 X_1 + w_2 X_2 + \cdots + w_n X_n.$$

   (a) If the estimators are unbiased, i.e. if $\beta_i = 0$ for all i, show that a linear combination $\hat{\mu}$ as above is unbiased if and only if $\sum w_i = 1$;

   (b) In the case when $\beta_i = 0$ for all i, show that an unbiased linear combination has minimum variance when the weights w_i are inversely proportional to the variances σ_i^2;

   (c) Show that the variance of $\hat{\mu}$ for optimal weights w_i is $\mathbf{V}(\hat{\mu}) = 1/\sum_i \sigma_i^{-2}$;

   (d) Next, consider the case where the estimators may be biased so we could have $\beta_i \neq 0$. Find the mean square error of the optimal linear combination obtained above, and compare its behaviour as $n \to \infty$ in the biased and unbiased case, when $\sigma_i^2 = \sigma^2, i = 1, \ldots, n$;

   (e) Same as (d), but for a general sequence σ_i^2.

4. Let X have density

$$f(x; \theta) = c(\theta) x^2 e^{-\theta x}, x > 0$$

   where $\theta > 0$ is assumed unknown.

   (a) Show that $c(\theta) = \theta^3/2$;

   (b) Show that $\tilde{\theta} = 2/X$ is an unbiased estimator of θ and find its variance;

   (c) Find the Fisher information $i(\theta)$ for the parameter θ and compare the variance of $\tilde{\theta}$ to the Cramér–Rao lower bound;

   (d) Let now $\mu = 1/\theta$ and show that $\hat{\mu} = X/3$ is an unbiased estimator of μ;

   (e) Find the variance of $\hat{\mu}$ and show that it attains the Cramér–Rao bound.

   Hint: Use standard properties of the Gamma function as in Question 1.