BS2 Problem Sheet 3

Question 4

The probability densities

$$f(x;\theta) = \frac{e^{\theta x - c(\theta)}}{1+x^4}, \quad \text{for } x \geq 0,$$

form an exponential family for $\theta \leq 0$ with log-normalising function

$$c(\theta) = \log\left(\int_0^\infty \frac{e^{\theta x}}{1+x^4} dx\right).$$

The canonical sufficient statistic is x and since this is an exponenential family the expectation and variance of X are

$$\mathbf{E}_\theta(X) = c'(\theta) \quad \text{and} \quad \mathbf{V}_\theta(X) = c''(\theta), \quad \text{for } \theta < 0. \qquad (*)$$

(b) By direct integration

$$\mathbf{E}_0(X) = \int_0^\infty \frac{xe^{-c(0)}}{1+x^4} dx = \frac{\pi}{4} e^{-c(0)} = 1/\sqrt{2},$$

since

$$e^{c(0)} = \int_0^\infty \frac{1}{1+x^4} dx = \frac{\pi}{4}\sqrt{2}.$$

From $(*)$ it follows that $\mathbf{E}_\theta(X)$ is non-decreasing since its derivative $c''(\theta)$ is the variance which is necessarily non-negative. It follows that $\mathbf{E}_\theta(X)$ cannot exceed $1/\sqrt{2}$, so that the likelihood equation has no solution when $x > 1/\sqrt{2}$.

(c) The derivative of $\log(f(x;\theta))$ with respect to θ is $x - c'(\theta)$ which is always positive when $x > 1/\sqrt{2}$. It follows that $f(x;\theta)$ has its maximum at $\theta = 0$ for these cases.

Parts (e) and (d) then follow similarly again with $\theta \leq 0$ but this time from first principles

$$\mathbf{E}_0(X) = \int_0^\infty \frac{xe^{-c(0)}}{1+x^2} dx = \frac{\pi}{4} e^{-c(0)} = \infty,$$

where

$$e^{c(0)} = \int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi}{2},$$

so that $\mathbf{E}_0(X)$ is not bounded and the likelihood equation always has a solution.