1. Consider a Wishart distributed 4×4 matrix $W \sim \mathcal{W}_4(n, \Sigma)$ where

$$\Sigma = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 3 & 15 & 5 & 9 \\ 1 & 5 & 2 & 3 \\ 2 & 9 & 3 & 6 \end{pmatrix} \text{ and } W = \begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{44} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{pmatrix}.$$

(a) Find the distribution of

$$W_{\{3,4\}} = \begin{pmatrix} W_{33} & W_{34} \\ W_{43} & W_{44} \end{pmatrix};$$

This is Wishart $W_2(n, \Sigma_{\{3,4\}})$ where

$$\Sigma_{\{3,4\}} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}.$$

(b) Find the conditional distribution of $W_{\{1,2\},\{3,4\}}$ given $W_{\{3,4\}}$, where

$$W_{\{1,2\},\{3,4\}} = (W_{13}, W_{14}, W_{23}, W_{24})^\top.$$

We first need to find the conditional covariance matrix

$$\Sigma_{\{1,2\}|\{3,4\}} = \begin{pmatrix} 1 & 3 \\ 3 & 15 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 5 & 9 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 5 \\ 2 & 9 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix},$$

where we have used that

$$\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2/3 \end{pmatrix}.$$

The conditional distribution of $(W_{13}, W_{14}, W_{23}, W_{24})^\top$ given $W_{\{3,4\}}$ is $\mathcal{N}_4(\xi, \Lambda)$. To find the expectation ξ we calculate

$$\begin{pmatrix} 1 & 2 \\ 5 & 9 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}^{-1} \begin{pmatrix} W_{33} & W_{34} \\ W_{43} & W_{44} \end{pmatrix} = \begin{pmatrix} W_{43}/3 & W_{44}/3 \\ W_{33} + W_{43} & W_{34} + W_{44} \end{pmatrix},$$

yielding

$$\xi^\top = (W_{43}/3, W_{44}/3\, W_{33} + W_{43}, W_{34} + W_{44}).$$

The covariance matrix becomes

$$\Sigma = \begin{pmatrix} W_{33}/3 & W_{34}/3 & 0 & 0 \\ W_{43}/3 & W_{44}/3 & 0 & 0 \\ 0 & 0 & W_{33} & W_{34} \\ 0 & 0 & W_{43} & W_{44} \end{pmatrix}.$$

2. Consider a sample $(X = x) = (X_1 = x_1, \ldots, X_n = x_n)$ from a normal distribution $\mathcal{N}(\theta, \phi)$.

   (a) Show that the likelihood function is

   $$L(\theta, \phi) \propto \phi^{-n/2} \exp\{-\frac{s}{2\phi} - \frac{n(\theta - \bar{x})^2}{2\phi}\}$$

   where $\bar{x} = n^{-1} \sum_i x_i$ and $S = \sum_i^n (x_i - \bar{x})^2$.
   Just write the joint density and use the usual decomposition of the sum of squares as

   $$\sum(x_i - \theta)^2 = \sum(x_i - \bar{x} + \bar{x} - \theta)^2 = \sum(x_i - \bar{x})^2 + n(\theta - \bar{x})^2.$$

   (b) Consider the (improper) prior distribution $\pi(\theta, \phi) \propto \phi^{-1}$ and show that the marginal posterior of θ is

   $$\pi(\theta \,|\, x) \propto \{s + n(\theta - \bar{x})^2\}^{-n/2}.$$

   The marginal posterior is

   $$\pi(\theta \,|\, x) \propto \int L(\theta, \phi)/\phi \, d\phi.$$

   Use the substitution $\xi = \phi^{-1}$ and the expression for the Gamma integral yields the result. Note this is in fact a Student's t-distribution, just scaled and shifted.

   (c) Show that the marginal posterior distribution of ϕ is

   $$\pi(\phi \,|\, x) \propto \phi^{-(n+1)/2} \exp\{-s/(2\phi)\}.$$

   The marginal posterior is

   $$\pi(\phi \,|\, x) \propto \int L(\theta, \phi)/\phi \, d\theta.$$

   Using the standard expression for the normal integral yields the result.

   (d) Show that the posterior density of $\gamma = \log \phi$ is

   $$\pi(\gamma \,|\, x) \propto \exp\{-\gamma(n-1)/2 - e^{-\gamma} s/2\}.$$

   Just integration by substitution, or standard transformation of variables.

   (e) Show that the posterior density of $\xi = \phi^{-1}$ is

   $$\pi(\xi \,|\, x) \propto \xi^{(n-3)/2} \exp\{-s\xi/2\}.$$

   Just integration by substitution, or standard transformation of variables.

(f) Find the posterior mode, mean, and median of ϕ, γ, ξ.

First the modes. For each density, take logarithms and differentiate to obtain

$$\check{\phi} = \frac{s}{n+1}, \quad \check{\gamma} = \log\frac{s}{n-1}, \quad \check{\xi} = \frac{n-3}{s}.$$

Note that they are different in the sense that $\log\check{\phi}$ is not equal to $\check{\gamma}$ and so on.

This is true for the means as well. Easiest first to look ξ which follows a $\chi^2(n-1)$, scaled by $1/s$. Thus $\bar{\xi} = \frac{n-1}{s}$.

The median of ξ is thus m_{n-1}/s, where m_{n-1} is the median in the $\chi^2(n-1)$ distribution. For the others, the median transforms correctly so the median of ϕ is s/m_{n-1} and the median of γ is $\log(s/m_{n-1})$.

The mean of γ is the mean of $-\log\xi$, where $\xi = 2U/s$ with U Gamma-distributed with shape parameter $\alpha = (n-1)/2$. Thus, since

$$\mathbf{E}(\log U) = \psi(\alpha)$$

where ψ is the digammafunction, we get

$$\mathbf{E}(\gamma \,|\, x) = \mathbf{E}(-\log\xi \,|\, x) = \log(s/2) - \psi\{(n-1)/2\}.$$

The mean of ϕ is similarly

$$\mathbf{E}(\phi \,|\, x) = \mathbf{E}(\xi^{-1} \,|\, x) = s/(n-3).$$

(g) Compare the marginal posterior densities with those obtained by Laplace approximation of the relevant integrals.

For the three last marginal distributions, the integral to be calculated *is* a normal integral, so is identical to its Laplace approximation. This is actually also true for the first integral: Maximizing the integrand yields

$$\phi^* = \frac{s + n(\theta - \bar{x})^2}{n-2}.$$

The second derivative of the log density is

$$\frac{n-2}{2\phi^2} - \frac{s + n(\theta - \bar{x})^2}{\phi^3}.$$

Inserting ϕ^* yields

$$j(\phi^*) = \frac{n-2}{\phi^{*2}}.$$

Inserting these into the Laplace integral yields

$$\pi(\theta \,|\, x) \approx\propto \phi^{*-(n+2)/2}\sqrt{1/\phi^*} = \phi^{*-n/2} \propto \{s + n(\theta - \bar{x})^2\}^{-n/2}.$$

Hence *this is also exact*!

(h) Use the Laplace approximation to derive an approximate expression for the density of $\eta = \theta - \gamma$.

I don't think there is anything much simpler than writing the joint posterior density of (η, ϕ) where $\eta = \theta - \log \phi$ as

$$-2 \log f(\eta, \phi \,|\, x) = 2g(\eta, \phi) = (n-2) \log \phi + s/\phi + n(\eta + \log \phi - \bar{x})^2/\phi.$$

Next integrating w.r.t. ϕ using Laplace's method. We then need to determine ϕ^*_η, maximizing the above expression for fixed η and find the second partial derivative of the above function w.r.t. ϕ at this maximum. Neither of these can be calculated explicitly, but must be calculated by Newton iteration.

If we let

$$r(\eta) = \left. \frac{\partial^2 g(\eta, \phi)}{\partial \phi^2} \right|_{\phi = \phi^*(\eta)}$$

the Laplace approximation becomes

$$f(\eta \,|\, x) \approx e^{-g(\eta, \phi^*_\eta)} \sqrt{\frac{2\pi}{r(\eta)}}.$$

Leonard and Hsu (2005), page 192–193, uses Lagrange multipliers, but it essentially amounts to the same thing.