1. Show that if $\Lambda \sim \Lambda(d, f_1, f_2)$ then

$$\Lambda \stackrel{\mathcal{D}}{=} \prod_{i=1}^{d} B_i$$

where $B_i$ are independent and follow Beta distributions with

$$B_i \sim \mathcal{B}\{(f_1 + 1 - i)/2, f_2/2)\}.$$

You may without proof use the fact that if $W \sim \mathcal{W}_d(f, \Sigma)$ and if $\Sigma_{12} = 0$, then $W_{1\,|\,2}$, $W_{12}W_{22}^{-1}W_{21}$, and $W_{22}$ are independent and Wishart distributed as

$$W_{1\,|\,2} \sim \mathcal{W}_r(f - s, \Sigma_{11}), \quad W_{12}W_{22}^{-1}W_{21} \sim \mathcal{W}_r(s, \Sigma_{11}), \quad W_{22} \sim \mathcal{W}_s(f, \Sigma_{22}).$$

*Hint:* Use induction after $d$ and exploit the identity

$$\det A = \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \det(A_{22}).$$

For an arbitrary (invertible) square $d \times d$ matrix $M$ write

$$\det M = \prod_{i=1}^{d} \frac{\det M_{[i]}}{\det M_{[i-1]}},$$

where $M_{[i]}$ contains the first $i$ rows and columns of $M$ so $M = M_{[d]}$ and $\det M_{[0]} = 1$.

Represent $\Lambda$ as

$$\Lambda = \frac{\det(W_1)}{\det(W_1 + W_2)}$$

with $W_j \sim \mathcal{W}_d(f_j, I_d), j = 1, 2$ and independent. We now get

$$\frac{\det W_{1[i]}}{\det W_{1[i-1]}} = \det(W_{1i|1\ldots,i-1}) = W_{1i|1\ldots,i-1},$$

which is distributed as $\chi^2(f_1 - i + 1)$ and independent of $W_{1[i-1]}$ using the partitioning result for the Wishart distribution. It can also be seen by realizing that $W_{1i|1\ldots,i-1}$ is the residual sum of squares from regressing $X_i$ on $X_1, \ldots, X_{i-1}$ . Similarly for $W_1 + W_2$. Thus

$$\Lambda = \frac{\det(W_1)}{\det(W_1 + W_2)} = \frac{\prod_{i=1}^{d} W_{1i|1\ldots,i-1}}{\prod_{i=1}^{d} W_{(1+2)i|1\ldots,i-1}} = \prod_{i=1}^{d} \frac{W_{1i|1\ldots,i-1}}{W_{(1+2)i|1\ldots,i-1}} = \prod_{i} B_i.$$

2. Show that if $W \sim \mathcal{W}_d(f, \Sigma)$, and if $\Sigma_{12} = 0$, then

$$U = W_{12} W_{22}^{-1} W_{21} \sim \mathcal{W}_r(s, \Sigma_{11}).$$

Further, $U$ and $W_{22}$ are independent.

*Hint:* Find first the conditional distribution of $U$ given $W_{22}$, and use that for any positive (semi)definite matrix $B$ there is a unique positive (semi)definite matrix $A$ so $B = A^2$. The matrix $A = B^{-1/2}$ is the *square root* of $B$. Identify the (conditional) distribution of rows in $W_{22}^{-1/2} W_{21}$.

The conditional distribution of $W_{12}$ given $W_{22} = w_{22}$ is

$$W_{12} \,|\, W_{22} = w_{22} \sim \mathcal{N}_{s\times r}(0, \Lambda)$$

with $\Lambda_{ij,kl} = \sigma_{ik} w_{jl}$.

Note that $w$ gives the covariance between columns, whereas $\Sigma$ yields that between rows. After post-multiplying with $W_{22}^{-1/2}$ we have

$$W_{12} W_{22}^{-1/2} \,|\, W_{22} = w_{22} \sim \mathcal{N}_{r\times s}(0, \Lambda)$$

with $\Lambda_{ij,kl} = \sigma_{ik}\delta_{jl}$, with $\delta_{jl} = 1$ if $j = l$ and 0 otherwise. Thus rows in $W_{12} W_{22}^{-1/2}$ (or columns in $W_{22}^{-1/2} W_{21}$) are independent and identically distributed as $\mathcal{N}_s(0, \Sigma_{11})$. Hence

$$U = W_{12} W_{22}^{-1/2} W_{22}^{-1/2} W_{21} = W_{12} W_{22}^{-1} W_{21} \sim \mathcal{W}_r(s, \Sigma_{11}).$$

3. Consider $U$ and $V$ independent and Beta distributed with

$$U \sim \mathcal{B}(\gamma, \delta), \quad V \sim \mathcal{B}(\gamma + 1/2, \delta),$$

where the Beta distribution has density

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1.$$

Show that

$$Z = \sqrt{UV} \sim \mathcal{B}(2\gamma, 2\delta).$$

This may be more tricky than it seems. It should be possible to derive this by simple integration, but it appears to be easier using moments: The (fractional) moments of $\mathcal{B}(\alpha, \beta)$ are

$$\mathbf{E}X^\lambda = \frac{\Gamma(\alpha+\lambda)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+\lambda)},$$

which is obtained as the ratio of normalizing constants for $\mathcal{B}(\alpha+\lambda,\beta)$ and $\mathcal{B}(\alpha,\beta)$. We thus have

$$\begin{aligned}\mathbf{E}(\sqrt{UV}^n) &= \mathbf{E}(U^{n/2})\mathbf{E}(V^{n/2}) \\ &= \frac{\Gamma(\gamma+n/2)\Gamma(\gamma+\delta)\Gamma\{\gamma+(n+1)/2\}\Gamma(\gamma+\delta+1/2)}{\Gamma(\gamma)\Gamma(\gamma+\delta+n/2)\Gamma(\gamma+1/2)\Gamma\{\gamma+\delta+(n+1)/2\}}.\end{aligned}$$

Use now *Legendre's formula*

$$\Gamma(x)\Gamma(x+1/2)=\Gamma(2x)\sqrt{\pi}2^{2x-1}$$

to obtain

$$\mathbf{E}(\sqrt{UV}^n)=\frac{\Gamma(2\gamma+n)\Gamma(2\gamma+2\delta)}{\Gamma(2\gamma)\Gamma(2\gamma+2\delta+n)}.$$

These are now recognized as the moments of $\mathcal{B}(2\gamma,2\delta)$. As a distribution on $(0,1)$ is determined by its moments, the result follows.

4. Let $\Lambda\sim\Lambda(d,f_1,f_2)$. Show that

   (a) If $d=1$:

   $$\frac{1-\Lambda}{\Lambda}\frac{f_1}{f_2}\sim F(f_2,f_1);$$

   $\Lambda(1,f_1,f_2)=\mathcal{B}(f_1/2,f_2/2)$. Hence this is the definition of $F$.

   (b) If $d=2$:

   $$\frac{1-\sqrt{\Lambda}}{\sqrt{\Lambda}}\frac{f_1-1}{f_2}\sim F\{2f_2,2(f_1-1)\};$$

   Use the first question to deduce that

   $$\Lambda\overset{\mathcal{D}}{=}UV$$

   where $V\sim\mathcal{B}\{f_1/2,f_2/2)\}$ and $U\sim\mathcal{B}\{(f_1-1)/2,f_2/2)\}$ are independent. Use the third question to deduce that then

   $$\sqrt{\Lambda}\sim\mathcal{B}(f_1-1,f_2).$$

   The result now follows as above.

   (c) If $f_2=1$:

   $$\frac{1-\Lambda}{\Lambda}\frac{f_1+1-d}{d}\sim F(d,f_1+1-d),$$

   Just use that $\Lambda(d,f_1,f_2)=\Lambda(f_2,f_1+f_2-d,d)$. The result now follows from (a).

   (d) If $f_2=2$:

   $$\frac{1-\sqrt{\Lambda}}{\sqrt{\Lambda}}\frac{f_1+1-d}{d}\sim F\{2d,2(f_1+1-d)\}.$$

   Just use that $\Lambda(d,f_1,f_2)=\Lambda(f_2,f_1+f_2-d,d)$. The result now follows from (b).