1. Let $X = (X_1, \ldots, X_n)$ be a sample from the Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ both unknown, i.e. the distribution with individual densities

$$f(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x}, \quad x > 0.$$

The canonical minimal sufficient statistic is $T = (S, C) = (\sum_i \log X_i, \sum_i X_i)$.

(a) Find the marginal density of C;
The sum of Gamma variables with known shape parameter is itself Gamma distributed so

$$f(c; \alpha, \beta) = \frac{\beta^{n\alpha} c^{n\alpha-1}}{\Gamma(n\alpha)} e^{-\beta c}.$$

(b) Show that for fixed α, C is sufficient for β;
When α is fixed, the joint density of $X_1, \ldots, X_n$ factorizes as

$$L(\alpha, \beta) = \frac{\beta^{n\alpha} \prod_i x_i^{\alpha-1}}{\Gamma(\alpha)^n} e^{-\beta c(x)} = h(\alpha, x) g(c(x), \alpha, \beta)$$

and Neyman's criterion yields the result.

(c) Find the conditional likelihood function for α;
Dividing the joint density with the marginal yields

$$L(\alpha \mid C = c) = \frac{\Gamma(n\alpha)}{\Gamma(\alpha)^n} \prod_i (x_i/c)^{\alpha-1},$$

(also showing that $(X_1/C, \ldots X_n/C)$ follows a Beta distribution).

(d) Find the profile likelihood function for α;
Maximizing the joint likelihood function over β yields $\hat{\beta}(\alpha) = n\alpha/c = \alpha/\bar{x}$ and hence

$$\hat{L}(\alpha) \propto (\alpha/\bar{x})^{n\alpha} \Gamma(\alpha)^{-n} e^{\alpha(s-n)}.$$

(e) Find the integrated likelihood for α when β is given a Gamma prior distribution with density

$$\pi(\beta) \propto \frac{b^a}{\Gamma(b)} \beta^{a-1} e^{-b\beta}.$$

We get

$$\begin{aligned}
\bar{L}(\alpha) &= \int L(\alpha, \beta) \pi(\beta)\, d\beta \\
&\propto e^{s\alpha} \Gamma(\alpha)^{-n} \int \beta^{n\alpha} e^{-\beta c} \frac{b^a}{\Gamma(b)} \beta^{a-1} e^{-b\beta} \\
&\propto e^{s\alpha} \Gamma(\alpha)^{-n} \Gamma(a + n\alpha)(b + c)^{-(n\alpha+a)}.
\end{aligned}$$

(f) Discuss inference for α when β is a nuisance parameter.
The conditional likelihood function eliminates the effect of the nuisance parameter and would be safe to use.

2. Consider $X_1 \sim \mathcal{N}(0,1)$ and define X_2 as

$$X_2 = \begin{cases} X_1 & \text{if } |X_1| > c \\ -X_1 & \text{otherwise.} \end{cases}$$

Determine c so that X_1 and X_2 are uncorrelated.
We get

$$\mathbf{E}(X_1X_2) = 2\int_c^\infty x^2\phi(x)\,dx - 2\int_0^c x^2\phi(x)\,dx.$$

Substituting $u = x^2$ we get

$$\begin{aligned}\int_c^\infty x^2\phi(x)\,dx &= \frac{1}{\sqrt{2\pi}}\int_c^\infty x^2 e^{-x^2/2}\,dx = \frac{1}{2\sqrt{2\pi}}\int_{c^2}^\infty ue^{-u/2}\frac{1}{\sqrt{u}}\,du \\ &= \frac{2^{3/2}\Gamma(3/2)}{2\sqrt{2\pi}}\{1-F_3(c)\} = \frac{1}{2}\{1-F_3(c)\}\end{aligned}$$

where F_3 is the distribution function of the χ^2-distribution with three degrees of freedom. Thus

$$\mathbf{E}(X_1X_2) = 1 - 2F_3(c^2)$$

and X_1 and X_2 are uncorrelated if

$$c = \sqrt{F_3^{-1}(1/2)} = 1.538172.$$

3. Let $X \sim \mathcal{N}_d(0, \sigma^2 I_d)$ where I_d is the $d \times d$ identity matrix and let O be an orthogonal $d \times d$ matrix, i.e. $O^\top O = OO^\top = I_d$. Show that $Y = OX \sim \mathcal{N}_d(0, \sigma^2 I_d)$.
We have that if $X \sim \mathcal{N}_d(\xi, \Sigma)$ then $AX \sim \mathcal{N}A\xi, A\Sigma A^\top)$. Hence with the above specification

$$OX \sim \mathcal{N}_d(O0, O\sigma^2 I_d O^\top) = \mathcal{N}_d(0, \sigma^2 OO^\top) = \mathcal{N}_d(0, \sigma^2 I_d).$$

4. Let $X = (X_1, X_2, X_3)$ be multivariate Gaussian $\mathcal{N}_3(\xi, \Sigma)$ with

$$\xi = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 2 & 2 \\ 4 & 2 & 5 \end{pmatrix}.$$

(a) Find the distribution of $X_1 + X_2$;
Since

$$(1,1)\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 8$$

we have

$$X_1 + X_2 \sim \mathcal{N}(2+3, 7) = \mathcal{N}(5, 8).$$

(b) Find the conditional distribution of X_3 given $X_1 = 0$;
We first marginalise to obtain the covariance matrix of (X_1, X_3) as

$$\Sigma_{13} = \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$

Then

$$\Sigma_{3|1} = 5 - 16/4 = 1,$$

so

$$X_3 \mid X_1 = 0 \sim \mathcal{N}\{-1 + \frac{1}{4}4(0-2), 1\} = \mathcal{N}(-3, 1).$$

(c) Find the concentration matrix $K = \Sigma^{-1}$;
We have to invert the covariance matrix to get

$$K = \Sigma^{-1} = \frac{1}{3}\begin{pmatrix} 6 & 3 & -6 \\ 3 & 4 & -4 \\ -6 & -4 & 7 \end{pmatrix}.$$

(d) Find the conditional distribution of (X_1, X_2) given $X_3 = 1$.
The concentration matrix is

$$K = \frac{1}{3}\begin{pmatrix} 6 & 3 \\ 3 & 4 \end{pmatrix}$$

so by inversion

$$\Sigma_{12|3} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -3 & 6 \end{pmatrix},$$

which could also have been found directly from the covariance matrix. The conditional expectation is easiest to calculate as

$$\xi_{12|3} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 4 \\ 2 \end{pmatrix}\{1 - (-1)\} = \frac{1}{5}\begin{pmatrix} 18 \\ 19 \end{pmatrix}.$$

(e) Find the conditional distribution of $X_1 + X_2$ given $X_3 = 1$.
We can now proceed as in (a) to find

$$X_1 + X_2 \mid X_3 = 1 \sim \mathcal{N}(37/5, 4/5).$$