1. Let $X_1, \dots, X_n$ be independent and identically normally distributed as $\mathcal{N}(\mu, \mu^2)$ with $\mu > 0$ being unknown. Thus the observations have *constant coefficient of variation* $\sqrt{\mathbf{V}(X)/\mathbf{E}(X)}$.

   (a) Show that $T = (U, V) = (\sum_i X_i/n, \sum_i X_i^2/n)$ is minimal sufficient;
   The likelihood function is

$$\begin{aligned} l(\mu) &= \log L(\mu) = c_1 - n \log \mu - (\sum_i x_i^2 - 2\sum_i x_i\mu + n\mu^2)/(2\mu^2) \\ &= c_2 - n \log \mu - nv/(2\mu^2) + nu/\mu \end{aligned}$$

   where c_1 and c_2 are constants. Three values of the likelihood function thus determines linear equations for c, u and v so (u, v) can be determined from the likelihood function:

$$\begin{aligned} -2l(1)/n &= c + v - 2u \\ -2l(2)/n &= c + 2\log 2 + v/4 - u \\ -2l(3)/n &= c + 2\log 3 + v/9 - 2u/3. \end{aligned}$$

   Hence (u, v) is equivalent to the likelihood function.

   (b) Show that $A = U/\sqrt{V}$ is ancillary;
   Clearly $Y = X/\mu$ is distributed as $\mathcal{N}(1, 1)$, independently of μ. But

$$A = U/\sqrt{V} = Z/\sqrt{W},$$

   where $(Z, W) = (\sum_i Y_i/n, \sum_i Y_i^2/n)$, hence its distribution cannot depend on μ.

   (c) Show that (U, V) is not complete;
   Let $\alpha = \mathbf{E}_\mu(A)$ which is independent of μ because of the above. Then, for $h(U, V) = U/\sqrt{V} - \alpha$ we have

$$\mathbf{E}_\mu\{h(U, V)\} = 0 \text{ for all values of } \mu.$$

   Hence (U, V) is not complete.

   (d) Discuss inference about μ.
   Inference about μ should then be based on the conditional distribution of U given A.

2. Let $X = (X_1, \ldots, X_n)$ be a sample of size n from the uniform distribution on the interval $(\psi - \lambda, \psi + \lambda)$:

$$f(x;\theta) = \frac{1}{2\lambda} \text{ for } \psi - \lambda < x < \psi + \lambda \text{ and 0 otherwise,}$$

where $\theta = (\psi, \lambda)$ with $-\infty < \psi < \infty$ and $\lambda > 0$ both unknown.

(a) Show that $(X^{(1)}, X^{(n)})$ is minimal sufficient;
The likelihood function is

$$L(\psi, \lambda) = \begin{cases} (2\lambda)^{-n} & \text{if } \psi - \lambda > x_{(1)} \text{ and } \psi + \lambda < x_{(n)} \\ 0 & \text{otherwise} \end{cases}$$

so $(X^{(1)}, X^{(n)})$ is clearly sufficient. But since $(X^{(1)}, X^{(n)})$ also can be inferred from the likelihood function it is also minimal sufficient.

(b) Show that the maximum likelihood estimator of θ is

$$\hat{\psi} = (X_{(1)} + X_{(n)})/2; , \quad \hat{\lambda} = (X_{(n)} - X_{(1)})/2;$$

The likelihood function is maximized when λ is minimized, subject to the constraint that the interval $(\psi - \lambda, \psi + \lambda)$ must contain $X_{(1)}$ and $X_{(n)}$. This minimum is attained when

$$\hat{\psi} = (X_{(1)} + X_{(n)})/2, \quad \hat{\lambda} = (X_{(n)} - X_{(1)})/2.$$

(c) Show that the distribution of $C = (X_{(n)} - X_{(1)})/2$ does not depend on ψ;
Let $Y = X - \psi$. The distribution of Y does not depend on ψ. Since

$$C = \hat{\lambda} = (X_{(n)} - X_{(1)})/2 = (Y_{(n)} - Y_{(1)})/2,$$

the distribution of C cannot depend on ψ.

(d) Let $(U, V) = ((X_{(1)} - \psi)/\lambda, (X_{(n)} - \psi)/\lambda)$ and show that the joint density of (U, V) is

$$f(u, v) = \begin{cases} n(n-1)(v-u)^{n-2}/2^n & \text{if } -1 < u < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hint: Find first $P(U \le u, V \le v)$.
Since $(X - \psi)/\lambda$ is uniformly distributed on the interval $-1 < x < 1$, we get for $-1 < u < v < 1$ that

$$P(U > u, V \le v) = \frac{(v-u)^n}{2^n}$$

and differentiation w.r.t. u and v yields the density required.

(e) Find the conditional density of $\hat{\psi} = (X_{(1)} + X_{(n)})/2$, given $C = c$.

Let $B = (U + V)/2$ and $A = (V - U)/2$. This linear transformation has determinant $1/2$ so the joint density of (a, b) is

$$f(a,b) = \begin{cases} 2n(n-1)a^{n-2} & \text{if } -1 < b - a < b + a < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The conditional density of B given $A = a$ is then found by keeping a constant so

$$f(b \,|\, a) \propto \begin{cases} 1 & \text{if } -1 < b - a < b + a < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now $C = \lambda A$ and $\hat{\psi} = \lambda B + \psi$ so $\hat{\psi}$ is uniformly distributed as:

$$f(\hat{\psi} \,|\, c) \propto \begin{cases} 1 & \text{if } -\lambda + c + \psi < \hat{\psi} < \psi + \lambda - c \\ 0 & \text{otherwise.} \end{cases}$$

(f) Discuss conditional inference for ψ when $\lambda = 1$ is known.

When λ is known, C is ancillary, so inference about ψ should be made in the conditional distribution of $\hat{\psi}$ given $C = c$ as derived above. If c is very close to one, this distribution is very narrow, as then $\hat{\psi} - \psi$ is conditionally uniform on $(c - 1, 1 - c)$, which is then a very small interval, and $\hat{\psi}$ is a very precise estimate. If on the other hand c is close to zero, the estimate $\hat{\psi}$ is very inaccurate and the corresponding confidence interval imprecise.

When λ is unknown, it still makes sense to condition on C, but now the conditional distribution of $\hat{\psi}$ given $C = c$ still depends on λ, so C does not form a cut.

3. Consider X and Y as independent Poisson random variables with $\mathbf{E}(X) = \gamma$ and $\mathbf{E}(Y) = \delta$, where $\gamma, \delta > 0$ are both unknown. We wish to find a similar test for equality of the two Poisson rates, i.e. the hypothesis $H_0 : \gamma = \delta$.

   (a) Show that under the null hypothesis, $C = X + Y$ is sufficient and complete;

   The joint probability mass function is

$$p(x, y; \gamma, \delta) = \frac{\gamma^x \delta^y}{x!y!} e^{-\gamma-\delta} = \frac{1}{x!y!} e^{x \log \gamma + y \log \delta - \gamma - \delta}.$$

   If $\gamma = \delta$ this is a one-parameter linear exponential family with $X + Y$ as the canonical sufficient statistic and $\theta = \log \gamma = \log \delta$ as canonical parameter. Thus $C = X + Y$ is sufficient and complete under the hypothesis.

   (b) Find the conditional distribution of X given $C = c$;

   Under the hypothesis, $C = X + Y$ is Poisson with mean $\gamma + \delta = 2\gamma = 2\delta$. Thus for $x + y = c$

$$p(x \,|\, C = c; \gamma) = \frac{p(x, y; \gamma)}{p(c; \gamma)} = \frac{c!}{x!c-x!} \frac{1}{2^c} = \binom{c}{x} \frac{1}{2^c}$$

   i.e. the conditional distribution is binomial.

   (c) Describe a similar test for the hypothesis H_0.

   A similar test for H_0 is now consctructed by rejecting for $X > l(c)$ or $X < -l(c)$ where $l(c)$ is the upper $1 - \alpha$ quantile in the binomial distribution for c trials with parameter $1/2$.