Some of these may be a bit too difficult ... The last question is easiest, but it uses results from the previous questions.

1. Show that if $\Lambda \sim \Lambda(d, f_1, f_2)$ then

$$\Lambda \stackrel{\mathcal{D}}{=} \prod_{i=1}^{d} B_i$$

where B_i are independent and follow Beta distributions with

$$B_i \sim \mathcal{B}\{(f_1 + 1 - i)/2, f_2/2)\}.$$

You may without proof use the fact that if $W \sim \mathcal{W}_d(f, \Sigma)$ and if $\Sigma_{12} = 0$, then $W_{1\,|\,2}$, $W_{12}W_{22}^{-1}W_{21}$, and W_{22} are independent and Wishart distributed as

$$W_{1\,|\,2} \sim \mathcal{W}_r(f - s, \Sigma_{11}), \quad W_{12}W_{22}^{-1}W_{21} \sim \mathcal{W}_r(s, \Sigma_{11}), \quad W_{22} \sim \mathcal{W}_s(f, \Sigma_{22}).$$

Hint: Use induction after d and exploit the identity

$$\det A = \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \det(A_{22}).$$

2. Show that if $W \sim \mathcal{W}_d(f, \Sigma)$, and if $\Sigma_{12} = 0$, then

$$U = W_{12}W_{22}^{-1}W_{21} \sim \mathcal{W}_r(s, \Sigma_{11}).$$

Further, U and W_{22} are independent.

Hint: Find first the conditional distribution of U given W_{22}, and use that for any positive (semi)definite matrix B there is a unique positive (semi)definite matrix A so $B = A^2$. The matrix $A = B^{-1/2}$ is the *square root* of B. Identify the (conditional) distribution of rows in $W_{12}W_{22}^{-1/2}$.

3. Consider U and V independent and Beta distributed with

$$U \sim \mathcal{B}(\gamma, \delta), \quad V \sim \mathcal{B}(\gamma + 1/2, \delta),$$

where the Beta distribution has density

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1 - x)^{\beta-1}, \quad 0 < x < 1.$$

Show that

$$Z = \sqrt{UV} \sim \mathcal{B}(2\gamma, 2\delta).$$

4. Let $\Lambda \sim \Lambda(d, f_1, f_2)$. Show that

   (a) If $d = 1$:
   $$\frac{1-\Lambda}{\Lambda}\frac{f_1}{f_2} \sim F(f_2, f_1);$$

   (b) If $d = 2$:
   $$\frac{1-\sqrt{\Lambda}}{\sqrt{\Lambda}}\frac{f_1-1}{f_2} \sim F\{2f_2, 2(f_1-1)\};$$

   (c) If $f_2 = 1$:
   $$\frac{1-\Lambda}{\Lambda}\frac{f_1+1-d}{d} \sim F(d, f_1+1-d),$$

   (d) If $f_2 = 2$:
   $$\frac{1-\sqrt{\Lambda}}{\sqrt{\Lambda}}\frac{f_1+1-d}{d} \sim F\{2d, 2(f_1+1-d)\}.$$