1. Let $X = (X_1, \ldots, X_n)$ be a sample from the Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ both unknown, i.e. the distribution with individual densities
$$f(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x}, \quad x > 0.$$
The canonical minimal sufficient statistic is $T = (S, C) = \sum_i \log X_i, \sum_i X_i)$.
   (a) Find the marginal density of C;
   (b) Show that for fixed α, C is sufficient for β;
   (c) Find the conditional likelihood function for α;
   (d) Find the profile likelihood function for α;
   (e) Find the integrated likelihood for α when β is given a Gamma prior distribution with density
$$\pi(\beta) \propto \frac{b^a}{\Gamma(b)} \beta^{a-1} e^{-b\beta}.$$
   (f) Discuss inference for α when β is a nuisance parameter.

2. Consider $X_1 \sim \mathcal{N}(0, 1)$ and define X_2 as
$$X_2 = \begin{cases} X_1 & \text{if } |X_1| > c \\ -X_1 & \text{otherwise.} \end{cases}$$
   (a) Show that $X_2 \sim \mathcal{N}(0, 1)$;
   (b) Determine c so that X_1 and X_2 are uncorrelated.

3. Let $X \sim \mathcal{N}_d(0, \sigma^2 I_d)$ where I_d is the $d \times d$ identity matrix and let O be an orthogonal $d \times d$ matrix, i.e. $O^\top O = OO^\top = I_d$. Show that $Y = OX \sim \mathcal{N}_d(0, \sigma^2 I_d)$.

4. Verify the matrix identities
$$K_{11}^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \tag{1}$$
and
$$K_{11}^{-1} K_{12} = -\Sigma_{12}\Sigma_{22}^{-1}, \tag{2}$$
where $K = \Sigma^{-1}$ is partitoned into blocks as
$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

5. Let $X = (X_1, X_2, X_3)$ be multivariate Gaussian $\mathcal{N}_3(\xi, \Sigma)$ with

$$\xi = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 2 & 2 \\ 4 & 2 & 5 \end{pmatrix}.$$

(a) Find the distribution of $X_1 + X_2$;

(b) Find the conditional distribution of X_3 given $X_1 = 0$;

(c) Find the concentration matrix $K = \Sigma^{-1}$;

(d) Find the conditional distribution of (X_1, X_2) given $X_3 = 1$

(e) Find the conditional distribution of $X_1 + X_2$ given $X_3 = 1$.