1. Consider a sample $(X_1 \dots, X_n) = (x_1, \dots, x_n)$ of independent observations from a canonical exponential family with density

$$f(x;\theta) = b(x)e^{\theta^\top t(x) - c(\theta)}.$$

The *reciprocal score* $R(\theta)$ is defined by

$$\mathcal{R}(\theta) = -\frac{\partial}{\partial\theta} L(\theta)^{-1}$$

i.e. the derivative of the reciprocal of the likelihood function rather than its logarithm.

(a) Show that

$$R(\theta) = L(\theta)^{-1}\{t(X) - \tau(\theta)\} = L(\theta)^{-1}S(\theta)$$

where $\tau(\theta) = \mathbf{E}_\theta\{t(X)\}$.

(b) Show that using Newton's method on the reciprocal score equation $R(\theta) = 0$ leads to the iteration

$$\theta \leftarrow \theta + \{v(\theta) + S(\theta)S(\theta)^\top\}^{-1}S(\theta).$$

(c) Compare this method to the method of scoring.

2. Let $X = (X_1, \dots, X_n)$ be a sample from the *Weibull* distribution with individual densities

$$f(x;\theta) = \theta x^{\theta-1}e^{-x^\theta} \text{ for } x > 0$$

where $\theta > 0$ is unknown.

(a) Find the score statistic and the likelihood equation;

(b) Show that if the likelihood equation has a solution, it must be the MLE;

(c) Describe the Newton–Raphson method for solving the likelihood equation;

(d) Describe the method of scoring for solving the likelihood equation;

(e) Can you think of other methods for solving the likelihood equation?

3. Consider a sample $X = (X_1, \ldots, X_n)$ from a normal distribution $\mathcal{N}(\mu, \mu^2)$, where $\mu > 0$ is unknown. This corresponds to the coefficient of variation $\sqrt{\mathbf{V}(X)}/\mathbf{E}(X)$ being known and equal to 1.

   (a) Find the score function for μ;

   (b) Show that the likelihood equation has a unique root $\hat{\mu}$ within the parameter space unless X_i are all equal to zero;

   (c) Show that the observed information at $\hat{\mu}$ is

   $$j(\hat{\mu}) = \frac{n}{\hat{\mu}^2} + \frac{\sum_i X_i^2}{\hat{\mu}^4},$$

   and use this to argue that the root $\hat{\mu}$ is indeed the MLE of μ;

   (d) Show that the Fisher information is equal to $(3n)/\mu^2$.

4. Let $X = (X_1, \ldots, X_n)$ be a sample from the Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ both unknown, i.e. the distribution with individual densities

   $$f(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x}, \quad x > 0.$$

   (a) Show that the score statistic for $\theta = (\alpha, \beta)$ is equal to

   $$S(\alpha, \beta) = \begin{pmatrix} \sum_i \log X_i + n \log \beta - n\psi(\alpha) \\ n\alpha/\beta - \sum_i X_i \end{pmatrix}$$

   where $\psi(\alpha) = D \log \Gamma(\alpha)$ is the *digamma* function;

   (b) Show that the method of scoring for θ leads to the iteration

   $$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leftarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \frac{1}{n\{\alpha\psi'(\alpha) - 1\}} \begin{pmatrix} \alpha & \beta \\ \beta & \beta^2\psi'(\alpha) \end{pmatrix} S(\alpha, \beta),$$

   where $\psi'(\alpha)$ is the *trigamma* function.

   (c) Consider a simpler iteration for solving the likelihood equation by first eliminating β from the equation.