1. This is essentially problem 5.1.a of Leonard and Hsu (1999).

   Consider a sample $(X = x) = (X_1 = x_1, \ldots, X_n = x_n)$ from a normal distribution $\mathcal{N}(\theta, \phi)$.

   (a) Show that the likelihood function is

   $$L(\theta, \phi) \propto \phi^{-n/2} \exp\{-\frac{s}{2\phi} - \frac{n(\theta - \bar{x})^2}{2\phi}\}$$

   where $\bar{x} = n^{-1} \sum_i x_i$ and $S = \sum_i^n (x_i - \bar{x})^2$.

   (b) Consider the (improper) prior distribution $\pi(\theta, \phi) \propto \phi^{-1}$ and show that the marginal posterior of θ is

   $$\pi(\theta \,|\, x) \propto \{s + n(\theta - \bar{x})^2\}^{-n/2}.$$

   (c) Show that the marginal posterior distribution of ϕ is

   $$\pi(\phi \,|\, x) \propto \phi^{-(n+1)/2} \exp\{-s/(2\phi)\}.$$

   (d) Show that the posterior density of $\gamma = \log \phi$ is

   $$\pi(\gamma \,|\, x) \propto \exp\{-\gamma(n-1)/2 - e^{-\gamma} s/2\}.$$

   (e) Show that the posterior density of $\xi = \phi^{-1}$ is

   $$\pi(\xi \,|\, x) \propto \xi^{(n-3)/2} \exp\{-s\xi/2\}.$$

   (f) Find the posterior mode, mean, and median of ϕ, γ, ξ.

   (g) Compare the marginal posterior densities with those obtained by Laplace approximation of the relevant integrals.

   (h) Use the Laplace approximation to derive an approximate expression for the density of $\eta = \theta - \gamma$.