Laplace's Method of Integration

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Consider an integral of form

$$I = \int_a^b e^{-\lambda g(y)} h(y)\, dy$$

where

1. λ is large;
2. $g(y)$ is a smooth function which has a local minimum at y^* in the interior of the interval (a, b);
3. $h(y)$ is smooth.

The integral can be the moment generating function of the distribution of $g(Y)$ when Y has density h, it could be a posterior expectation of $h(Y)$, or just an integral.

When λ is large, the contribution to this integral is essentially entirely originating from a neigbourhood around y^*.

We formalize this by Taylor expansion of the function g around y^*:

$$g(y) = g(y^*) + g'(y^*)(y - y^*) + g''(y^*)(y - y^*)^2/2 + \cdots$$

Since y^* is a local minimum, we have $g'(y^*) = 0$, $g''(y^*) > 0$, and thus

$$g(y) - g(y^*) = g''(y^*)(y - y^*)^2/2 + \cdots$$

If we further approximate $h(y)$ linearly around y^* we get

$$\begin{aligned} I &= \int_a^b e^{-\lambda g(y)} h(y)\, dy \\ &\approx e^{-\lambda g(y*)} h(y^*) \int_{-\infty}^{\infty} e^{-\lambda g''(y^*)(y-y^*)^2/2}\, dy \\ &\quad + e^{-\lambda g(y*)} h'(y^*) \int_{-\infty}^{\infty} (y - y^*) e^{-\lambda g''(y^*)(y-y^*)^2/2}\, dy \\ &= e^{-\lambda g(y*)} h(y^*) \sqrt{\frac{2\pi}{\lambda g''(y^*)}} + 0. \end{aligned}$$

We have exploited that we know the integral and expectation of a Gaussian density with concentration $g''(y^*)\lambda$. The approximation is typically very accurate and satisfies

$$\begin{aligned} I &= \int_a^b e^{-\lambda g(y)} h(y)\, dy \\ &= e^{-\lambda g(y*)} h(y^*) \sqrt{\frac{2\pi}{\lambda g''(y^*)}} \left\{1 + O\left(\frac{1}{\lambda}\right)\right\} = A \left\{1 + O\left(\frac{1}{\lambda}\right)\right\} \end{aligned}$$

meaning that the relative error

$$\frac{I - A}{A}$$

is $O(\lambda^{-1})$ and thus remains bounded for $\lambda \to \infty$, *even when multiplied with λ*.

Consider the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

and recall that for integers λ we have

$$\Gamma(\lambda + 1) = \lambda!$$

We get

$$\Gamma(\lambda + 1) = \int_0^\infty t^{\lambda} e^{-t}\, dt.$$

Substituting $y = t/\lambda$ and letting $g(y) = y - \log y$ we get

$$\Gamma(\lambda + 1) = \lambda \int_0^\infty (\lambda y)^{\lambda} e^{-\lambda y}\, dy = \lambda^{\lambda+1} \int_0^\infty e^{-\lambda g(y)}\, dy.$$

To use Laplace's method we differentiate twice and get

$$g'(y) = 1 - 1/y, \quad g''(y) = 1/y^2$$

so that $y^* = 1$, $g(y^*) = 1$ and $g''(y^*) = 1$. Laplace's method now yields

$$\begin{aligned}\Gamma(\lambda+1) &= \lambda^{\lambda+1} e^{-\lambda g(y*)} \sqrt{\frac{2\pi}{\lambda g''(y^*)}} \left\{1 + O\left(\frac{1}{\lambda}\right)\right\} \\ &= \lambda^{\lambda+1/2} e^{-\lambda} \sqrt{2\pi} \left\{1 + O\left(\frac{1}{\lambda}\right)\right\}\end{aligned}$$

which is known as *Stirling's formula*.

By expanding the function g further, the error of approximation can be improved for a constant function h so that

$$\begin{aligned}\tilde{I} &= \int_a^b e^{-\lambda g(y)}\,dy \\ &= e^{-\lambda g(y*)}\sqrt{\frac{2\pi}{\lambda g''(y^*)}}\left\{1+\frac{5\rho_3^*-3\rho_4^*}{24\lambda}+O\left(\frac{1}{\lambda^2}\right)\right\},\end{aligned}$$

where

$$\rho_3^* = \frac{g^{(3)}(y^*)}{\{g''(y^*)\}^{3/2}}, \quad \rho_4^* = \frac{g^{(4)}(y^*)}{\{g''(y^*)\}^2}.$$

In this fashion we can also get *Stirling's improved formula* as

$$\Gamma(\lambda+1) = \lambda^{\lambda+1/2} e^{-\lambda} \sqrt{2\pi} \left\{ 1 + \frac{1}{12\lambda} + O\left(\frac{1}{\lambda^2}\right) \right\}$$

which is remarkably accurate, even for rather small values of λ, as this table of $\log \Gamma(\lambda+1)$ shows:

λ	Exact	Stirling	Improved
2	0.6931472	0.6518048	0.6926268
4	3.1780538	3.1572615	3.1778807
8	10.6046029	10.5941899	10.6045527
16	30.6718601	30.6666508	30.6718456
32	205.1681995	205.1668957	205.1681970

Alternatively, if the variation of h around y^* is not neglible, or a more accurate approximation is desired, one can incorporate h in g as

$$\tilde{g}_\lambda(y) = g(y) - \frac{1}{\lambda} \log h(y)$$

and get the approximation

$$\begin{aligned} I &= \int_a^b e^{-\lambda g(y)} h(y)\, dy \\ &= \int_a^b e^{-\lambda \tilde{g}_\lambda(y)}\, dy \\ &= e^{-\lambda \tilde{g}_\lambda(\tilde{y}_\lambda)} \sqrt{\frac{2\pi}{\lambda \tilde{g}''_\lambda(\tilde{y}_\lambda)}} \left\{ 1 + \frac{5\tilde{\rho}_3 - 3\tilde{\rho}_4}{24\lambda} + O\left(\frac{1}{\lambda^2}\right) \right\}, \end{aligned}$$

where now $\tilde{y}_\lambda$ maximizes $\tilde{g}_\lambda(y)$, and other quantities are similarly defined.

The multivariate case is completely analogous. Here we again write

$$g(y) = g(y^*) + \frac{\partial g(y^*)}{\partial y}(y - y^*) + (y - y^*)^\top \frac{\partial^2 g(y^*)}{\partial y \partial y^\top}(y - y^*)/2 + \cdots$$

and exploit that the vector of partial derivatives $\frac{\partial g(y^*)}{\partial y}$ must vanish, whereby

$$\begin{aligned} I &= \int_B e^{-\lambda g(y)} h(y)\, dy \\ &= e^{-\lambda g(y^*)} h(y^*) \int_{\mathcal{R}^d} e^{-\lambda (y-y^*)^\top \frac{\partial^2 g(y^*)}{\partial y \partial y^\top}(y-y^*)/2 + \dots}\, dy \\ &= e^{-\lambda g(y^*)} h(y^*)(2\pi/\lambda)^{d/2} \left| \frac{\partial^2 g(y^*)}{\partial y \partial y^\top} \right|^{-1/2} \left\{ 1 + O\left(\frac{1}{\lambda}\right) \right\}. \end{aligned}$$