Patterns of Thiele

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April 23, 2002

Abstract

This is a translation into English of T. N. Thieles paper on 'number-patterns' (Thiele 1874) where he uses properties of integer algebraic equations to produce beautiful pictures, thus representing an early precursor to such aspects of modern mathematics as algebraic geometry and the visual representation of fractals (Mandelbrot 1983; Peitgen and Richter 1986).

1 Introduction

T. N. Thiele (1838–1910) was a Professor of Astronomy at the University of Copenhagen who made pioneering contributions to statistics and actuarial science, in particular through his work on the method of least squares (Thiele 1880, 1889) and his work on halfinvariants (cumulants)(Thiele 1889, 1899); see Hald (1981) for an overview and Lauritzen (2002) for a commented collection of his most important statistical work in English translation.

Thiele was interested in many aspects of mathematics and its application, but his main enthusiasm was no doubt the mathematics of computation. He was in particular utterly fascinated by everything associated with numbers. He wrote an article on the axiomatization of numbers (Thiele 1886), although this seems not to have had any lasting impact. Gram (1910) mentions that Thiele advocated, almost mystically, the sequence $1, 4, 9, 16, 25, 35, 49, \ldots$ of squares of integers as dividing the life of a human into natural periods, so these were the birthdays lending themselves to special celebration rather than those divisible by 10.

Below is a translation of a small paper which Thiele gave at a conference of general scientists in Copenhagen, held in 1873. It is concerned with the use of complex numbers to produce pretty pictures. One of his students', the high school teacher Otto Larsen, writes (Larsen 1954) that Thiele had used these patterns to produce a fire screen in his home and that the tiles on the floor in the Danish insurance company Hafnia (founded by Thiele in 1873) were laid according to one of these patterns.

2 Thiele: On patterns of numbers

Next Dr. Thiele gave a lecture "On the patterns of numbers":

After an introduction where the relation between mathematics and creative art was discussed, the speaker showed a collection of drawings supporting his point that it is not only so that all geometric lines necessarily are beautiful and conversely that any beautiful "line" or collection of lines in a piece of art must satisfy various demands of a mathematical nature, but indeed there also exists a form of art in which mathematics itself, without the supplementing and veiling dress of a capricious imagination, can produce complete pictures which hardly could be described as being stiff and boring.

The art in question is that of mosaic or more precisely the drawing of patterns for weaving, embroidery etc., where the picture is produced by assigning different colours to mutually similar (congruent) small pieces of which a surface is composed. From a mathematical point of view this becomes a geometric application of the theory of integers. But, to be applicable to the geometry of the plane, the theory must first be generalised to include such integer numbers which are not real.

It is easily seen that the integers always must correspond to individuals which each is determined by its corresponding integer number, and that the plane, to ensure that only the integers are important, must be divided into small meshes with position determined by complex numbers. This is the only way in which the representation in itself can exclude such numbers which are not (or are not considered as) integer. If one assumes that $a + bi$, where $i = \sqrt{-1}$, is to be considered integer exactly when both a and b are integers, the corresponding geometric application becomes a plane divided into small squares of the same size. However, Gauss' representation of the complex numbers, by points with abscissa a and ordinate b, is not so suited for this purpose, as one is tempted to give rational numbers a significance similar to the integers.

If, for example, a given mesh on a piece of canvas is denoted by the number 0, then the neighbouring meshes would be denoted by $+1$, $+i$, -1 and $-i$. If one then counts a meshes from zero over $+1$ and then b meshes in the same direction as the location of $+i$ relative to 0, one will reach the mesh with position determined by the general integer $a + bi$. In this way every integer number determines a mesh and every mesh determines an integer number.

The theory of integer numbers, the extended as well as the simple, differs from proper mathematics in the sense that its problems are not generally either solvable for all numbers or for no numbers, but the situation for any general problem has essential differences, depending on the numbers entering into the problem. And these differences place numbers close to each other in the strongest possible contrast. This gives a character to pictures made by number theory, the patterns of numbers, which is completely different from usual geometric figures. For any problem which must be solved in terms of integers containing a single arbitrary element, a number-pattern can be made by assigning different colours to the meshes corresponding to this element depending on whether the problem has a solution when the arbitrary element is set to the value in question. The problem "general divisibility of numbers" corresponds to a pattern in which all meshes representing prime numbers are given the same colour (white), whereas the divisible numbers are marked with a different colour (black). Such a figure of prime numbers in the form of a pattern was then distributed to the members of the audience.

Most of the other patterns shown referred partly to the divisibility of numbers with a given integer, partly to the "congruence" of degree 2, or the undetermined equation $y^2 - cx = d$. Some examples of these are shown here in photolitographic reprint (Figure 1). In the three patterns on top, the 0 square is placed in the bottom left corner and marked with a dot in its centre; the bottom horisontal row of small squares is then corresponding to the non-negative integers, the first vertical column of squares to the numbers $0, i, 2i, 3i$ and so on. The squares marked with dots represents the numbers divisible by $14 + 1i$, 19 and $17 + 8i$ respectively. However, the completely coloured squares which form the conspicuous patterns represent the numbers

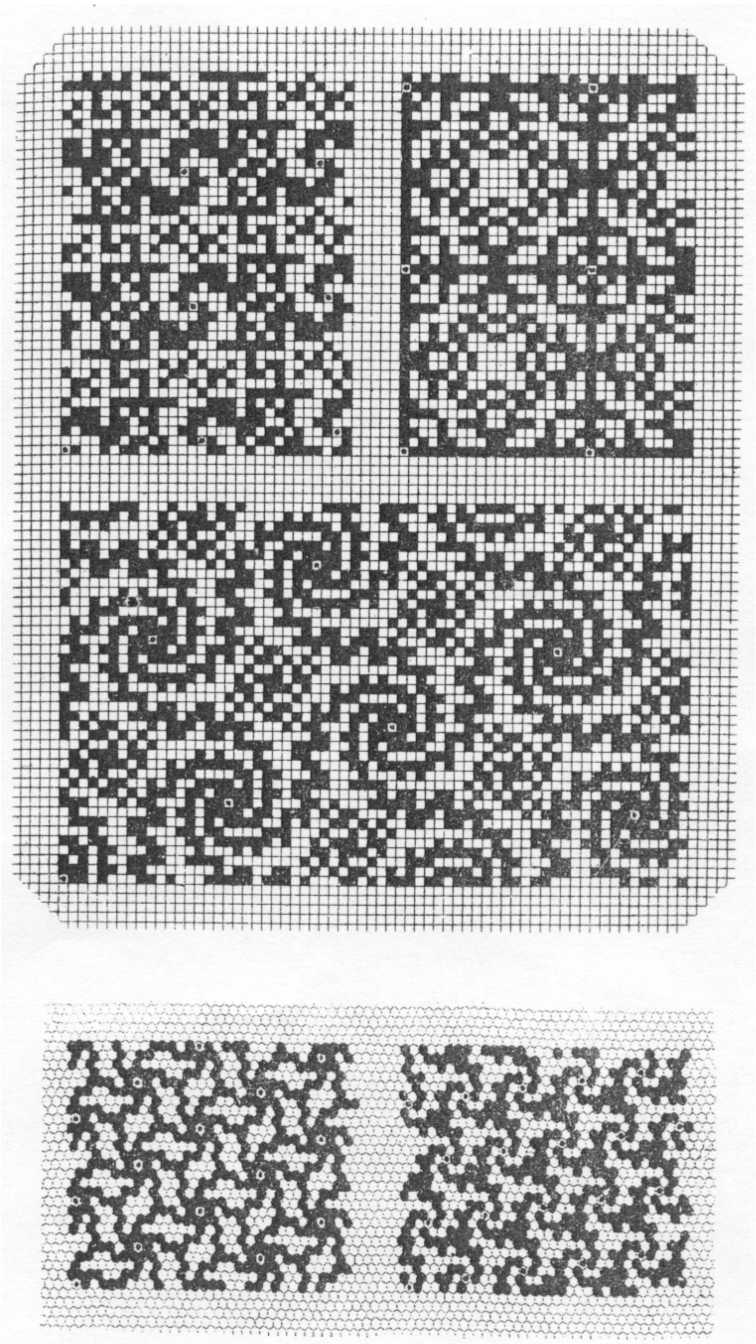

Figure 1: Reproduction of the figure displayed in Thiele (1874).

d for which the equations

$$\begin{aligned}
&y^2 - (14 + 1i)x = d \\
&y^2 - 19x = d \\
&y^2 - (17 + 8i)x = d
\end{aligned}$$

have solutions with integer values of x and y (assuming that d is not divisible by the values substituted for c in $y^2 - cx = d$).

All these patterns display an apparent arbitrariness in their details, as if they had been the product of an uncontrolled imagination and did not express the strictest regularity. But looking at such a pattern at a distance sufficiently large for the eye not to be caught by the smallest details, a richness of lines and relationships in these figures is revealed in terms of lighter and darker shadings that bear witness of the true origin of the figures, which in the strictest possible sense must be regarded as "severe style"[1].

Some of these lines and relationships can be deduced easily from the mathematical form of the corresponding problems, for example the logarithmic spirals, straight lines and circles, but the accompanying figures show other lines, in particular the prominent S-shapes, which for the speaker appear as mathematical riddles without an obvious solution.

Finally the speaker mentioned that the extension of the notion of integer numbers to complex numbers of the form, integer plus integer times the number i, by no means is canonical. The extension can be made in many ways, also in such a way that $i = \sqrt{-1}$ does not become an integer. In particular it was mentioned that considering the number $\sqrt[3]{-1}$ as the simplest integer one could use $a + b\sqrt[3]{-1}$, where a and b are integers, as a different general common expression for the complex numbers without contradiction or significant deviation from older number theory. The geometric application of this representation would demand that the plane is covered with a mesh of regular hexagons. The two patterns at the bottom of Figure 1 were drawn on this basis using a natural print of a bobbinet. The pattern to the left corresponds to the number $5+4\sqrt[3]{-1}$; the meshes with dots show the numbers which are divisible with this number whereas the coloured meshes show the values of d which make the equation

$$y^2 - (5 + 4\sqrt[3]{-1})x = d$$

[1] The expression "severe style" probably refers to the first phase of classical Greek sculpture, prevailing in the period 500–300 B.C. *Translator's remark.*

solvable for integer values of x and y of the same form $a+b\sqrt[3]{-1}$. The pattern to the left corresponds in the same way to the number $6+1\sqrt[3]{-1}$. The arrangement of these patterns are resembling the arrangement of the other patterns as closely as possible: 0 at the bottom left corner, and the positive integers in the bottom horisontal row. De 6 meshes which are neighbours to 0 are here 1, $\sqrt[3]{-1}$, $-1+\sqrt[3]{-1}=\sqrt[3]{1}$, -1, $-\sqrt[3]{-1}$ and $1-\sqrt[3]{-1}$.

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