Specification of Homophily in Actor-oriented Network Models

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using data collected by Vanina Torló

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Homophily

Homophily, the attraction of like to like, is a fundamental characteristic of human (and other) relations (Lazarsfeld & Merton, 1954; McPherson, Smith-Lovin & Cook, 2001). It is mostly theorized for binary or categorical attributes (sex, race, ....).

This presentation is about how to express homophily for continuous (or numerical ordinal) variables in statistical network models, focusing on (longitudinal) actor-oriented models.
Multilevel: actor – dyad

Homophily is a concept relating the actor level to the dyadic level, exemplifying the *multilevel nature* of network analysis, where concepts and measures of different types of units meet each other.

Usual expressions

Consider a numerical variable $V$.

A usual way to express tendencies toward homophily is by letting probabilities of tie existence or creation depend on the absolute difference between sender $i$ and receiver $j$,

$$| v_i - v_j |.$$

This has been proposed for many models (e.g., ERGM: Lusher et al., 2013; SAOM: Snijders, van de Bunt & Steglich, 2010; $p_2$: van Duijn et al., 2004; other: Powell et al., AJS 2005).
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As an alternative, it has been proposed to make the probability of tie existence or creation dependent on the ego $\times$ alter interaction

$$v_i v_j \text{ or } (v_i - \bar{v})(v_j - \bar{v}).$$
Criticism of the usual expressions

For numerical variables, however, these models are quite poor approximations.
Criticism of the usual expressions

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It is helpful to conceptualize the probabilities of tie creation and existence as being dependent on a preference or attraction function indicating how strongly one social actor is attracted to another, in dependence on their values of the attribute; for some attributes this function will be unimodal (single-peaked), for others it may be monotone (‘the higher the better’), or even bimodal (repulsive).
Diverse principles of attraction

For a given numerical attribute $V$ and relations $X$, there may be various different principles of attraction:

1. homophily: attraction to others with the same value;
2. aspiration: attraction to others with a high value;
3. conformity: attraction to others with the average/normative value;
4. and all this may be modified by the other effects included in the model.

Stokman and Vieth (2004, ‘What attracts us when to whom?’) also mention repulsion; this is left out of consideration here.

These principles will usually be combined.
Modeling attraction in SAOMs: absolute difference

In Stochastic Actor-Oriented Models, let us call the part of the evaluation function depending on the actor covariate $V$ the *attraction function*.

For the absolute difference, the attraction function is

$$f_1(v_j) = \beta_1 |v_i - v_j| .$$

$(\beta_1 < 0)$
Modeling attraction in SAOMs: absolute difference

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For the absolute difference, the attraction function is

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($\beta_1 < 0$)

top must be at $v_j = v_i$; not quite flexible
Modeling attraction in SAOMs: ego $\times$ alter interaction

For the representation by the ego $\times$ alter interaction, we also need the two main effects, and the attraction function is

$$f_2(v_j) = \beta_1 v_i + \beta_2 v_j + \beta_3 v_i v_j$$
Modeling attraction in SAOMs: ego × alter interaction

For the representation by the ego × alter interaction, we also need the two main effects, and the attraction function is

\[ f_2(v_j) = \beta_1 v_i + \beta_2 v_j + \beta_3 v_i v_j \]

not quite a representation of homophily
Modeling attraction in SAOMs: better model

A combination of attraction to different values can be mathematically expressed by a quadratic function of $v_i$ and $v_j$, because this allows expressing a shifted maximum by a linear term.

To include the homophily principle, it is convenient to express the optimum as a linear function of ego’s value, $v_i^{\text{max}} = \gamma_2 + \gamma_3 v_i$.

This leads to

$$f(v_j) = \gamma_1 (v_j - \gamma_3 v_i - \gamma_2)^2$$

$$= -2 \gamma_1 \gamma_2 v_j + \gamma_1 v_j^2 - 2 \gamma_2 \gamma_3 v_i + \gamma_1 \gamma_3^2 v_i^2 - 2 \gamma_1 \gamma_3 v_i v_j + \gamma_1 \gamma_2^2.$$
Modeling attraction in SAOMs: better model

Turning this into a function that is linear in all parameters yields

\[ f_3(v_j) = \beta_1 v_j + \beta_2 v_j^2 + \beta_3 v_i + \beta_4 v_i^2 + \beta_5 v_i v_j. \]
Parameter interpretation

For the function

\[ f_3(v_j) = \beta_1 v_j + \beta_2 v_j^2 + \beta_3 v_i + \beta_4 v_i^2 + \beta_5 v_i v_j : \]

1. The function is unimodal if \( \beta_2 < 0 \).

2. Higher values for ego go with attraction toward higher values of alter if \( \beta_5 > 0 \).

3. The optimum is obtained, for a given \( v_i \) and \( \beta_2 < 0 \), when

\[
v_i^{\text{max}} = \frac{\beta_1 + \beta_5 v_i}{-2\beta_2}.
\]

When this is outside the range of \( V \), the function will be increasing or decreasing over the entire range.
Expectations

This means that we expect:

1. unimodality: $\beta_2 < 0$ (effect $V^2$ alter)
2. homophily: $\beta_5 > 0$ (effect $V$ alter $\times$ $V$ ego)
3. if there is an element of aspiration:

$$\frac{\beta_1 + \beta_5 \bar{V}}{-2\beta_2} > \bar{V}$$

where $\bar{V}$ is the mean of $V$: on average, the attraction is to values higher than the mean.

For $\beta_2 < 0$, this is the same as $\beta_1 + (\beta_5 + 2\beta_2)\bar{V} > 0$.

For centered $V$, this is $\beta_1 > 0$ (effect $V$ alter)
Note that these interpretations and expectations are formulated without thinking of the rest of the model, and accordingly the expectations may be off if the rest of the model already has implications (e.g., by correlation of effects) for how ties depend on the values of $V$ for tie senders and receivers.
Example: Vanina Torló’s students

International MBA program in Italy; 75 students; 3 waves distributed over one year.

Two dependent networks:

1. *Friendship*: meaningful relations outside program context.
2. *Advice*: help, support on program-related tasks.

Relevant covariate: *Achievement*: average exam grades.
## Descriptives

<table>
<thead>
<tr>
<th></th>
<th>Friendship</th>
<th></th>
<th>Advice</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T_1$</td>
<td>$T_2$</td>
<td>$T_3$</td>
<td>$T_1$</td>
<td>$T_2$</td>
<td>$T_3$</td>
</tr>
<tr>
<td>Av. degree</td>
<td>9.9</td>
<td>9.2</td>
<td>9.3</td>
<td>4.1</td>
<td>4.9</td>
<td>4.5</td>
</tr>
<tr>
<td>Reciprocity</td>
<td>0.58</td>
<td>0.54</td>
<td>0.57</td>
<td>0.29</td>
<td>0.33</td>
<td>0.33</td>
</tr>
<tr>
<td>Transitivity</td>
<td>0.44</td>
<td>0.40</td>
<td>0.38</td>
<td>0.24</td>
<td>0.24</td>
<td>0.26</td>
</tr>
</tbody>
</table>

**Graph:**

A bar graph showing the distribution of grades across different time points ($T_1$, $T_2$, and $T_3$). The x-axis represents grades ranging from 20 to 30, and the y-axis represents frequencies ranging from 0% to 35%. Each bar is color-coded to indicate the time point: $T_1$ (black), $T_2$ (gray), and $T_3$ (white).
## Results

<table>
<thead>
<tr>
<th>Effect</th>
<th>par.</th>
<th>(s.e.)</th>
<th>par.</th>
<th>(s.e.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Friendship</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>outdegree (density)</td>
<td>−2.676</td>
<td>(0.178)</td>
<td>−3.225</td>
<td>(0.261)</td>
</tr>
<tr>
<td>reciprocity</td>
<td>1.696</td>
<td>(0.145)</td>
<td>0.641</td>
<td>(0.202)</td>
</tr>
<tr>
<td>transitive triplets</td>
<td>0.254</td>
<td>(0.024)</td>
<td>0.310</td>
<td>(0.046)</td>
</tr>
<tr>
<td>transitive recipr. triplets</td>
<td>−0.179</td>
<td>(0.034)</td>
<td>−0.104</td>
<td>(0.081)</td>
</tr>
<tr>
<td>indegree - popularity</td>
<td>0.046</td>
<td>(0.011)</td>
<td>0.052</td>
<td>(0.007)</td>
</tr>
<tr>
<td>outdegree - popularity</td>
<td>−0.065</td>
<td>(0.014)</td>
<td>−0.040</td>
<td>(0.031)</td>
</tr>
<tr>
<td>outdegree - activity</td>
<td>0.001</td>
<td>(0.005)</td>
<td>0.013</td>
<td>(0.012)</td>
</tr>
<tr>
<td>gender alter</td>
<td>0.017</td>
<td>(0.074)</td>
<td>0.008</td>
<td>(0.104)</td>
</tr>
<tr>
<td>gender ego</td>
<td>−0.123</td>
<td>(0.081)</td>
<td>−0.203</td>
<td>* (0.100)</td>
</tr>
<tr>
<td>same gender</td>
<td>0.233</td>
<td>(0.078)</td>
<td>−0.004</td>
<td>(0.095)</td>
</tr>
<tr>
<td>same natio</td>
<td>0.286</td>
<td>(0.102)</td>
<td>0.371</td>
<td>* (0.147)</td>
</tr>
<tr>
<td>grades alter</td>
<td>−0.028</td>
<td>(0.024)</td>
<td>0.112</td>
<td>** (0.035)</td>
</tr>
<tr>
<td>grades squared alter</td>
<td>−0.028</td>
<td>* (0.009)</td>
<td>−0.027</td>
<td>† (0.014)</td>
</tr>
<tr>
<td>grades ego</td>
<td>−0.097</td>
<td>** (0.025)</td>
<td>−0.037</td>
<td>(0.032)</td>
</tr>
<tr>
<td>grades squared ego</td>
<td>−0.019</td>
<td>* (0.008)</td>
<td>0.010</td>
<td>(0.010)</td>
</tr>
<tr>
<td>grades ego × alter</td>
<td>0.034</td>
<td>** (0.011)</td>
<td>0.044</td>
<td>** (0.016)</td>
</tr>
</tbody>
</table>

...continued...
## Results (continued)

<table>
<thead>
<tr>
<th>Effect</th>
<th>Friendship</th>
<th>par.</th>
<th>(s.e.)</th>
<th>Advice</th>
<th>par.</th>
<th>(s.e.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>advice</td>
<td>1.563***</td>
<td>(0.247)</td>
<td>—</td>
<td>1.628***</td>
<td>(0.265)</td>
<td>—</td>
</tr>
<tr>
<td>friendship</td>
<td>—</td>
<td>—</td>
<td>0.660**</td>
<td>(0.209)</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>reciprocity with advice</td>
<td>0.335</td>
<td>(0.249)</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>reciprocity with friendship</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>indegree advice popularity</td>
<td>−0.044***</td>
<td>(0.011)</td>
<td>—</td>
<td>−0.024*</td>
<td>(0.011)</td>
<td>—</td>
</tr>
<tr>
<td>indegree friendship popularity</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>outdegree advice activity</td>
<td>−0.030†</td>
<td>(0.016)</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>outdegree friendship activity</td>
<td>—</td>
<td>—</td>
<td>−0.053***</td>
<td>(0.012)</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
Attraction function for grades

The effect of grades on friendship is important.
Joint Wald test: \( \chi^2_5 = 33.8, \ p < 0.0001. \)

The effect of grades on advice also is significant, but less strongly:
Joint Wald test: \( \chi^2_5 = 16.9, \ p < 0.001. \)

Attraction function of friendship based on grade is

\[
f(v_j) = -0.028 v_j - 0.028 v_j^2 \]
\[
- 0.097 v_i - 0.019 v_i^2 \]
\[
+ 0.034 v_i v_j,
\]

where \( v = \text{grade} \) \(- 26.1.\)
Attraction point

Main expectations are borne out: $\hat{\beta}_2 < 0$, $\hat{\beta}_5 > 0$; note $\hat{\beta}_1 < 0$ (n.s.).

The maximum is attained for

$$v_{i}^{\text{max}} = \frac{\hat{\beta}_1 + \hat{\beta}_5 v_i}{-2\hat{\beta}_2} = \frac{(-0.028) + 0.034 v_i}{-2 \times (-0.028)} = -0.5 + 0.6v_i,$$

indeed increasing with $v_i$;

the intercept $\hat{\beta}_1 / (-2\hat{\beta}_2) = -0.5$ has s.e. $= 0.4$ (delta method), not significantly negative.
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the intercept $\hat{\beta}_1 / (-2\hat{\beta}_2) = -0.5$ has s.e. $= 0.4$ (delta method), not significantly negative.

How should we interpret that the plot shows generally lower values for higher $v_i$ (ego)?
How is this reflected in observed friendships?

<table>
<thead>
<tr>
<th>grade alter</th>
<th>≤ 23</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
<th>≥ 29</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>grade ego</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>≤ 23</td>
<td>.06</td>
<td>.25</td>
<td>.14</td>
<td>.36</td>
<td>.13</td>
<td>.06</td>
<td>.01</td>
<td>9.9</td>
</tr>
<tr>
<td>25</td>
<td>.05</td>
<td>.13</td>
<td>.12</td>
<td>.26</td>
<td>.25</td>
<td>.16</td>
<td>.03</td>
<td>17.6</td>
</tr>
<tr>
<td>26</td>
<td>.03</td>
<td>.08</td>
<td>.12</td>
<td>.29</td>
<td>.25</td>
<td>.19</td>
<td>.03</td>
<td>15.9</td>
</tr>
<tr>
<td>27</td>
<td>.03</td>
<td>.11</td>
<td>.11</td>
<td>.31</td>
<td>.23</td>
<td>.15</td>
<td>.06</td>
<td>15.8</td>
</tr>
<tr>
<td>28</td>
<td>.01</td>
<td>.05</td>
<td>.15</td>
<td>.33</td>
<td>.27</td>
<td>.10</td>
<td>.08</td>
<td>13.1</td>
</tr>
<tr>
<td>≥ 29</td>
<td>.03</td>
<td>.09</td>
<td>.14</td>
<td>.24</td>
<td>.22</td>
<td>.22</td>
<td>.06</td>
<td>19.5</td>
</tr>
</tbody>
</table>

Rowwise proportions of ties depending on grades of alter, given ego’s grade; and average outdegrees (‘total’); added for waves 2 and 3.

Distribution gradually, slightly, shifting upward. No trend in outdegrees.
How is the fit for observed friendships by grade?

Comparison observed - modeled dependence on grades: study goodness of fit by \textit{sienaGOF}.

Set of auxiliary statistic is defined as average number of ties, for egos with a given grade, to others with grades (respectively) 20-23, 24, 25, 26, 27, 28, and 29-30, added for the second and third waves (7 × 7 combinations of values).

Overall Mahalanobis combination $p = 0.045$. This is acceptable; see plot next page.
Students with higher grades do not have lower outdegrees. But look again at the selection table / attraction function.
Students with higher grades do not have lower outdegrees. But look again at the selection table / attraction function.

Lower curves for higher grades of ego should not be interpreted as a revealed preference for fewer friends!

Number of ties determined not only by attraction function, but also by distribution of ties at the previous wave, ‘supply’ of others with given grades, and other network influences. Main other network influences are reciprocation and transitivity.
Network embeddedness as a function of grade

Left: reciprocated degree as function of grade.
Right: number of others at directed distance two as function of grade.
Those with higher grades have more reciprocated ties and (especially) more open networks $\sim$ more candidates for transitive closure. These two positional characteristics work positively on their outdegrees. This is counteracted by the lower attraction functions. The result is a moderate increase of outdegrees as a function of grade.

Outdegree as function of grade.
Comparison with other attraction functions

The quadratic ego and alter effects are significant; this shows that the model with the absolute difference attraction function and that with the ego $\times$ alter attraction function for friendship as a function of grades fit worse than the quadratic model.

Contribution of grades to the evaluation function for friendship.
Left: linear interaction specification; right: similarity specification.
The attraction function for advice and grades

The estimated attraction function of advice based on grade is

\[ f(v_j) = 0.112 v_j - 0.027 v_j^2 \]
\[ - 0.037 v_i + 0.010 v_i^2 \]
\[ + 0.044 v_i v_j . \]

where again \( v = \text{grade} - 26.1 \).
The value for which the optimum is obtained is estimated as

\[ v_{i}^{\text{max}} = \frac{\hat{\beta}_1 + \hat{\beta}_5 v_i}{-2\hat{\beta}_2} = \frac{0.112 + 0.044 v_i}{-2 \times (-0.027)} = 2.1 + 0.8 v_i. \quad (2) \]

Standard errors now are, respectively, 0.6 for \( \hat{\beta}_1 / (-2\hat{\beta}_2) \), and 0.3 for \( \hat{\beta}_5 / (-2\hat{\beta}_2) \).

The group norm for advice accordingly is estimated as \( 2.1 + 0.8 \bar{v} = 28.2 \), significantly higher than average grade: a combination of aspiration with homophily.
Example

Attraction functions for friendship and advice

Selection functions for grades on friendship (left) and advice (right).
Discussion

- Tendencies toward homophily may and will coexist with aspirations toward high (low) values and with tendencies toward a common (average) value.
- Specifications that have been in use in statistical network modeling for expressing homophily on numerical actor covariates are too rigid (absolute difference) or no good representations of homophily (ego $\times$ alter interaction). Briefly, they are often misspecifications.
- A quadratic specification of the dependence on ego’s and alter’s value offers an important improvement.
- New effect egoSqX implemented in RSiena.
Discussion – continued

- This implies the recommendation to use five parameters to represent the effect of a numerical monadic actor variable on probabilities of tie existence and tie change. This seems like a lot, but is more meaningful for theoretical modeling than absolute difference or cross-product interaction.

- Three-parameter sub-model is possible, but will be non-linear, which leads to technical complications.

- For actor variables with moderate effects on the network, the extra parameters (quadratic effects) make little difference.

- For actor variables with strong effects on the network, the extra parameters (quadratic effects) may make a large difference.
Discussion – continued

- Homophily component can be tested by the ego-actor interaction in this five-parameter model (‘four control effects’).
- Aspiration component can be tested by the alter effect if the variable is centered.
- This model can resolve interpretation problems in cases where ‘negative homophily’ (repulsion) seemed to be found when using an absolute difference or simple interaction model.
- The value for alter where the optimum is assumed (dependent on ego’s value) is an interesting descriptive feature.
- The example offers an interesting perspective on the distinction between the evaluation function and a preference function.