Equivalence Concepts for Social Networks

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Outline

Block modeling

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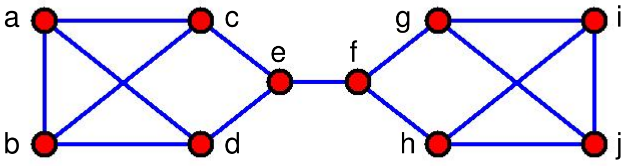
Lorrain and White (1971) defined that
nodes *a* and *b* are structurally equivalent,
if they relate to other nodes in the same way.

The Borgatti-Everett Network

The following is a network

proposed by Borgatti and Everett (1991).

Which nodes are structurally equivalent?



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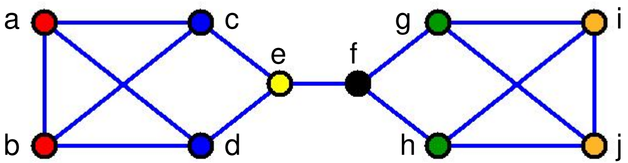

There are 6 colorings / equivalence classes

Image matrix

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Usually, the vertices have to be rearranged so that each color indicates a set of successive nodes; then the adjacency matrix shows a *block structure*.

Image matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & . & 1 & 0 & 0 \\ 0 & 0 & 1 & . & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The adjacency matrix has a block structure: all blocks are either all – 0 or all – 1.

Adjacency matrix

$$\left(\begin{array}{cc|cc|c|c|cc|cc} . & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & . & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & . & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & . & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & . & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & . & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & . & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & . & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & . & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & . \end{array}\right)$$

Approximate structural equivalence

For most empirically observed networks, hardly any nodes are structurally equivalent.
However, there may be groups of nodes that are *approximately structurally equivalent*.

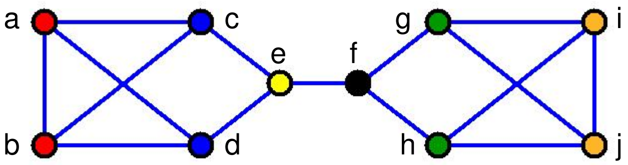
This is elaborated by defining the elements of the image matrix as the *proportion* of ties in the corresponding block of the adjacency matrix;
and striving for an image matrix with elements all of which are as close as possible to either 0 or 1.

As an exercise, you can run
"Operations – Blockmodeling" in Pajek
for Doreian's data set of 14 political actors,
and find approximate structural equivalence classes.

Uncheck the "short report" option,
and ask for 4 classes.
In the output, `com` means 'complete'.

Other equivalences

Here is the Borgatti and Everett (1991) network again:



This is the structural equivalence coloring.

Do you see other possibilities of equivalence?

Other equivalences

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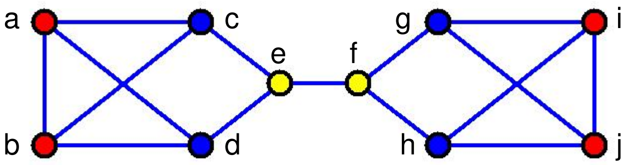

What about this coloring?

Doesn't is seem also a good representation of equivalence?

Regular equivalence

A coloring is a *regular equivalence*

(Sailer 1978; White and Reitz 1983)

if vertices with the same color

also have neighbors of the same color.

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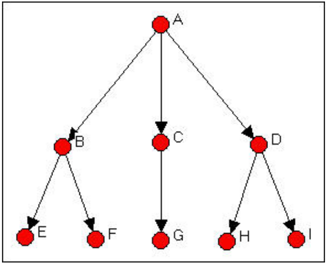
This definition is a nice mathematical representation
of the sociological concept of *role*:
the color / role determines
to which other colors / roles you should be tied;
what is required is being tied to *some* actors in this role,
not to *all* actors of this role.



Wasserman and Faust (1994)
give the following example.

You can look in Hanneman's
text (Section 15)
for further examples.

One graph can have many different colorings
that all are regular equivalences!

Stochastic equivalence

The classical concepts of equivalence in networks
can be applied to cases of approximate equivalence
by maximizing some measure of adequacy,
that measures how well the observed block structure
corresponds to what would be predicted
in the case of exact equivalence.
All nodes are classified in one of the classes.

Probability models provide another way
to express the deviations between observations and
the idealized concept of ("exact") equivalence.

For a probability distribution of the ties in a graph,
a coloring is a *stochastic equivalence*
(Fienberg and Wasserman, 1981)
if nodes with the same color have
the same *probability distribution* of ties with other nodes.

More formally:
the probability distribution of the graph must
remain the same when equivalent nodes are exchanged.
Such a distribution is called a *stochastic block model*.

The stochastic block model is a kind of
Latent Structure Analysis (LSA).
The basic idea of LSA, proposed by Lazarsfeld & Henry (1968),
is that there exist latent (i.e. unobserved) variables such that
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The *structural model* then specifies the latent variables and the *measurement model* specifies how the observations depend on the latent variables.
LSA has been extended to measurement models that specify not conditional independence, but more generally also allow simple, restricted, types of dependence.

The stochastic block model is a latent structure model where the latent structure is the node coloring,
which has to be recovered from the observed network;

for each pair of nodes *i* and *j*, the colors of these nodes determine the probability of a tie
or (for valued / multivariate networks)
of a certain tie configuration between *i* and *j*;
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This is a ‘rough’ type of network model,
which is useful for bringing out the global structure.

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1-blocks on diagonal, 0-blocks off-diagonal:

high within-group density,

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Blockmodeling is, however, much more general:
any difference between probabilities of ties
within and between groups is permitted.

Example of blockmodeling using Pajek:

From

`http://vlado.fmf.uni-lj.si/pub/networks/course/blockmodels.pdf`

pages 8–16.

Literature

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