Part A Statistics HT 2004 **Exercises 4** Dr. Gesine Reinert

1. When X, Y, Z are i.i.d. $\mathcal{N}(0, \theta)$, the distance U from the origin to the point (X, Y, Z) has the *Maxwell distribution*

$$f_U(u; \theta) \propto \frac{u^2}{\theta^{\frac{3}{2}}} e^{-\frac{u^2}{2\theta}}; \quad u > 0.$$

   A statistic $T = T(X_1, \ldots, X_n)$ is called *asymptotically efficient* if its asymptotic variance is equal to the Cramer-Rao lower bound. Show that the statistic $T = \frac{1}{3n} \sum_{i=1}^n U_i^2$ is asymptotically efficient in estimating θ.
   Hint: Write $U^2 = X^2 + Y^2 + Z^2$, where X, Y, Z are i.i.d. $\mathcal{N}(0, \theta)$. Use this to calculate mean and variance of T. You may use results from lectures about mean and variances of chisquare-distributed random variables.

2. Using the Neyman-Pearson Lemma, construct a uniformly most powerful test of the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_1 : \mu < \mu_0$ based on a random sample from a Poisson distribution with unknown mean μ. What would the corresponding results be if the alternative hypothesis were $H_1 : \mu > \mu_0$?
   Now suppose that the alternative hypothesis is $H_1 : \mu \neq \mu_0$. Calculate the test statistic in this case. If the null hypothesis is $\mu = 5$, the number in the sample is 50 and the sum of the observations is 319, use the asymptotic theory to test the null hypothesis against the general alternative $\mu \neq 5$.

3. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with mean $\frac{1}{\lambda}$. Prove that

$$P\left(b / \sum_{i=1}^n X_i < \lambda \right) = 1 - \alpha,$$

   where $1 - \alpha = \frac{1}{\Gamma(n)} \int_b^\infty y^{n-1} e^{-y} \, dy$. Let $x_1, x_2, \ldots, x_n$ be observed valued in a sample and let $c = \sum_{i=1}^n x_i$. Deduce that $(b/c_n, \infty)$ is a $100(1-\alpha)\%$ confidence interval for λ.
   For the same data it is required to test $\lambda = \lambda_0$ against the alternative hypothesis $\lambda > \lambda_0$. Show that the p-value for the likelihood ratio test satisfies $p = \frac{1}{\Gamma(n)} \int_0^{\lambda_0 c_n} y^{n-1} e^{-y} \, dy$. Deduce that if $p > \alpha$ then $\lambda_0 \in (b/c_n, \infty)$.

4. An experiment conducted to check the variability in explosion times of detonators intended to detonate simultaneously resulted in the following times to detonation in microseconds:
   2.689, 2.677, 2.675, 2.691, 2.698, 2.694, 2.702,
   2.698, 2.706, 2.692, 2.691, 2.681, 2.700, 2.698.
   Assume that the data come from a normally distributed random sample.
   (a) The specification is that the standard deviation $\sigma \leq 0.007$; is it met? Use a test of significance at level $\alpha = 0.05$.
   (b) Is there any evidence against the hypothesis that the mean explosion time is 2.7 microseconds? Carry out a test of significance at level $\alpha = 0.05$, assuming that the true standard deviation is $\sigma = 0.007$.
   (c) Now test the same hypothesis but do not assume that you know the true underlying variance.