Part A Statistics HT 2004 **Exercises 2** Dr. Gesine Reinert

1. Let $X_1, \ldots, X_n$ be independent and identically $\mathcal{N}(\mu, \sigma^2)$-distributed. Let
$$\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i \quad \text{and } S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2.$$
For each case, find a function of $(\bar{X}, S^2, \mu, \sigma^2)$ having the following distributions.
a) standard normal $\mathcal{N}(0,1)$
b) $\chi^2(n-1)$
c) $\chi^2(n)$
d) $t(n-1)$.
Derive the moment-generating function of S^2 and deduce that $ES^2 = \sigma^2$ and $Var S^2 = 2\sigma^4/(n-1)$.

2. Let $X_1, X_2, \ldots, X_n$ be independent having χ^2 distributions with parameters $r_1, r_2, \ldots, r_n$, respectively. Show that
$$Y = \sum_{i=1}^n X_i$$
has the χ^2-distribution with $\sum_{i=1}^n r_i$ degrees of freedom.
Conclude that, if $X_1, \ldots, X_n$ are independent normal random variables with means $\mu_1, \ldots, \mu_n$ and variances $\sigma_1^2, \ldots, \sigma_n^2$, respectively, then
$$Y = \sum_{i=1}^n \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2$$
has the χ^2_n-distribution.

3. A random vector $\mathbf{X}$ is said to have the *multivariate normal distribution* with parameters μ and Σ if its density is
$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{det\Sigma}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)}, \quad \mathbf{x} \in \mathbf{R}^n,$$
where Σ is an $n \times n$ symmetric matrix, assumed to be positive definite (i.e. $\mathbf{x}^t\Sigma\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$), called the *covariance matrix*, and μ is an n-vector, called the *mean vector*.
Suppose that $\mathbf{X}$ has the multivariate normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\sigma^2 I_n$, where I_n is the $n \times n$ identity matrix. Let Q be an $n \times n$ symmetric, idempotent (i.e. $Q^2 = Q$) matrix of rank r, and put
$$Y = \frac{1}{\sigma^2}\mathbf{X}^T Q \mathbf{X}.$$
a) Show that for sufficiently small t the m.g.f. of Y is
$$M_Y(t) = |det(I_n - 2tQ)|^{-\frac{1}{2}}.$$
b) Writing Q as NAN^T, where N is an orthogonal matrix whose columns are the eigenvectors of Q, deduce that Y has the χ^2_r-distribution.

4. Show that if X and Y are independent exponential random variables with mean 1, then $\frac{X}{Y}$ follows an F-distribution. Identify the degrees of freedom.