2. Normal sampling theory

From the central limit theorem: normal distribution plays central role

Recall: $X \sim \mathcal{N}(\mu, \sigma^2)$ if p.d.f.

$$f(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

for $-\infty < x < \infty$; alternatively, m.g.f.

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

X has $E(X) = \mu, Var(W) = \sigma^2$

$X \sim \mathcal{N}(\mu, \sigma^2)$ then

$$X = \mu + \sigma Z$$

where $Z \sim \mathcal{N}(0, 1)$

Theorem 1 *Suppose* $X_1, \ldots, X_n$ *independent,*

$X_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$ *for* $j = 1, \ldots, n$

Put $Y = \sum_{j=1}^n \alpha_j X_j$ *Then*

$$Y \sim \mathcal{N}(\sum_{j=1}^n \alpha_j \mu_j, \sum_{j=1}^n \alpha_j^2 \sigma_j^2)$$

Proof.

$$\begin{aligned}
M_Y(t) &= E\left(e^{tY}\right) \\
&= E \prod_{j=1}^n e^{t\alpha_j X_j} \\
&= \prod_{j=1}^n M_{X_j}(t\alpha_j) \quad (indep.) \\
&= \prod_{j=1}^n \exp\left(\alpha_j \mu_j t + \frac{\sigma_j^2 \alpha_j^2 t^2}{2}\right) \\
&= \exp\left(t \sum_{j=1}^n \alpha_j \mu_j + t^2 \sum_{j=1}^n \frac{\sigma_j^2 \alpha_j^2}{2}\right).
\end{aligned}$$

Example:

$X_1, \ldots, X_n$ i.i.d. $\mathcal{N}(\mu, \sigma^2)$ for $j = 1, \ldots, n$

$\alpha_j = 1/n$ for $j = 1, \ldots, n$

$$Y = \bar{X} = \frac{1}{n} \sum_{j=1}^{n} X_j \sim \mathcal{N}(\mu, \sigma^2/n)$$

The χ^2 distribution

Suppose $Z_1, \ldots, Z_r$ are i.i.d. $\mathcal{N}(0,1)$, and put

$$Y = Z_1^2 + Z_2^2 + \cdots + Z_r^2$$

The Y is said to have a χ_r^2-**distribution**, or a χ^2 *distribution with r degrees of freedom*

is the same as Gamma(α, λ) distribution with $\alpha = r/2$ and $\lambda = 1/2$

Theorem 2 *The p.d.f. of Y is*

$$f_Y(y) \;=\; \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{r/2}} y^{\frac{r}{2}-1} e^{-y/2}$$

for $0 \le y < \infty$

the m.g.f. is $M_Y(t) = (1 - 2t)^{-r/2}$ *for* $-\infty < t < 1/2$.

In particular

$$E(Y) = r$$

and

$$Var(Y) = 2r$$

Proof: Exercise.

Independence of $\bar{X}$ and S^2

$X_1, \ldots, X_n$ i.i.d. $\mathcal{N}(\mu, \sigma^2)$

Often estimate μ by $\bar{X}$, and σ^2 by

$$S^2 = \frac{1}{n-1} \sum_{j=1}^{n} (X_j - \bar{X})^2$$

Theorem 3 *If $X_1, \ldots, X_n$ i.i.d. $\mathcal{N}(\mu, \sigma^2)$ then $\bar{X}$ and S^2 are independent,*

$$\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Proof.

Write $X_j = \mu + \sigma Z_j$, where $Z \sim \mathcal{N}(0,1)$

$$\bar{X} = \mu + \sigma \bar{Z}$$

$$\frac{S^2}{\sigma^2} = \frac{1}{n-1} \sum_{j=1}^{n} (Z_j - \bar{Z})^2$$

p.d.f. of $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ is

$$f_{\mathbf{Z}}(\mathbf{z}) = (2\pi)^{-n/2} \exp\left(-\mathbf{z}^T\mathbf{z}/2\right)$$

for $\mathbf{z} \in \mathbf{R}^n$

Change of variable: $\mathbf{Y} = Q\mathbf{Z}$ where

Q is orthogonal, $Q^TQ = I, detQ = \pm 1$

first row of Q is identical to $n^{-1/2}$

Then

$$\mathbf{y}^T\mathbf{y} = \mathbf{z}^TQ^TQ\mathbf{z} = \mathbf{z}^T\mathbf{z}$$

Jacobian is 1 in absolute value, so

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-n/2} \exp\left(-\mathbf{y}^T\mathbf{y}/2\right)$$

so $Y_1, \ldots, Y_n \sim \mathcal{N}(0, 1)$, i.i.d. and

$$Y_1 = \sqrt{n}\bar{Z}$$

so

$$\begin{aligned}\sum_{j=1}^{n} (Z_j - \bar{Z})^2 &= \sum_{j=1}^{n} Z_j^2 - n(\bar{Z})^2 \\ &= \sum_{j=1}^{n} Y_j^2 - Y_1^2 \\ &= \sum_{j=2}^{n} Y_j^2\end{aligned}$$

As Y_1 and $Y_2, \ldots, Y_n$ are independent:

$\bar{Z}$ and $\sum_{j=1}^{n}(Z_j - \bar{Z})^2$ are independent,

$$\bar{Z} \sim \mathcal{N}(0, 1/n)$$

$$\sum_{j=1}^{n} (Z_j - \bar{Z})^2 \sim \chi^2_{n-1}$$

transform back to $X_1, \ldots, X_n$: gives the assertion.

Note: $E\bar{X} = \mu$ and $E(n-1)S^2/\sigma^2 = (n-1)$, so $ES^2 = \sigma^2$; S^2 is unbiased

Remark: Transformation $X_j = \mu + \sigma Z_j$ gives that the distribution of $\frac{S^2}{\sigma^2}$ does not depend on μ or σ^2

Similarly can show:

$$\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}, \qquad \frac{\sqrt{n}(\bar{X}-\mu)}{S}$$

are functions of $Z_1, Z_2, \ldots, Z_n$ alone, so their distribution cannot depend on μ or σ^2

Student's t-distribution

Let $X \sim \mathcal{N}(0,1)$ and $Y \sim \chi^2_r$, independent

The distribution of

$$T = \frac{X}{\sqrt{Y/r}}$$

is called **(Student's) t-distribution with** r **degrees of freedom**; $T \sim t_r$.

Theorem 4 *The p.d.f. of t_r is*

$$f_T(t) = \frac{\Gamma((r+1)/2)}{\Gamma(r/2)}(r\pi)^{-1/2}(1+t^2/r)^{-(r+1)/2},$$

for $-\infty < t < \infty$.

Proof Transformation formula. The joint p.d.f. of X and Y is

$$f_{X,Y}(x,y) = C_r e^{-x^2/2} y^{\frac{r}{2}-1} e^{-y/2},$$

for $-\infty < x < \infty, y \geq 0$; with

$$C_r = (\sqrt{2\pi}\Gamma(r/2)2^{r/2})^{-1}$$

and $t = x\left(\frac{y}{r}\right)^{-1/2}$ so

$$f_{T,Y}(t,y) = f_{X,Y}(t(y/r)^{1/2}, y)(y/r)^{1/2}$$

giving

$$\begin{aligned} f_T(t) &= r^{-1/2}C_r \int_0^\infty \exp\left\{-y\left(1+t^2/r\right)/2\right\} \\ &\quad y^{\frac{r+1}{2}-1}\,dy \end{aligned}$$

Substitute $u = y\left(1+t^2/r\right)/2$ to obtain

$$\begin{aligned} f_T(t) &= r^{-1/2}C_r\left(\frac{2}{1+t^2/r}\right)^{\frac{r+1}{2}} \\ &\quad \int_0^\infty e^{-u}u^{\frac{r+1}{2}-1}\,du \\ &= r^{-1/2}C_r\left(\frac{2}{1+t^2/r}\right)^{\frac{r+1}{2}}\Gamma((r+1)/2). \end{aligned}$$

Special cases

$r = 1$: $f_T(t) = (\pi(1+t^2))^{-1}$: Cauchy distribution

$r \to \infty$: $f_T(t) \to \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$
approximately standard normal

Theorem 5 *If $X_1, \ldots, X_n$ i.i.d. $\mathcal{N}(\mu, \sigma^2)$ then*

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

Proof:

$$\begin{aligned} T &= \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \times \frac{\sigma}{S} \\ &= \frac{V}{\sqrt{Y/(n-1)}} \end{aligned}$$

with $V = \sim \mathcal{N}(0,1)$, $Y = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, independent from Theorem 3

Fisher's F-distribution

Let $V \sim \chi^2_q$, $Y \sim \chi^2_r$, independent: The distribution of

$$W = \frac{V/q}{Y/r}$$

is called **(Fisher's) F-distribution with** (q, r) **degrees of freedom**,

$$W \sim F_{q,r}$$

Fact: The $F_{q,r}$-distribution has as density

$$f_W(w) \;=\; \frac{\Gamma((q+r)/2)}{\Gamma(q/2)\Gamma(r/2)} \left(\frac{q}{r}\right)^{q/2} \frac{w^{\frac{q}{r}-1}}{\left(1+\frac{q}{r}w\right)^{(q+r)/2}}$$

for $w \geq 0$

From the definition follows

Theorem 6 *Suppose, independently,* $X_1, \ldots, X_n$ *are i.i.d.* $\mathcal{N}(\mu_1, \sigma_1^2)$, $Y_1, \ldots, Y_m$ *are i.i.d.* $\mathcal{N}(\mu_2, \sigma_2^2)$, *and let*

$$S_1^2 = S_{XX}^2 = \frac{1}{n-1} \sum_{j=1}^{n} (X_j - \bar{X})^2$$

$$S_2^2 = S_{YY}^2 = \frac{1}{m-1} \sum_{j=1}^{m} (Y_j - \bar{Y})^2$$

and

$$W = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

Then $W \sim F_{n-1,m-1}$

Some properties:

If $X \sim F_{q,r}$ then $\frac{1}{X} \sim F_{r,q}$

If $X \sim t_r$ then $X^2 \sim F_{1,r}$

To calculate EW for $r \geq 3$:

If $Y \sim \chi^2_r$ then

$$E\left(\frac{1}{Y}\right) = E\int_{-\infty}^{0} e^{tY}\,dt = \int_{-\infty}^{0}(1-2t)^{-r/2}dt = \frac{1}{r-2}$$

If $V \sim \chi^2_q$ then $EV = q$

By independence follows

$$EW = E\left(\frac{V/q}{Y/r}\right) = \frac{r}{r-2}$$

tends to 1 for $r \to \infty$

For $r = 2$, use that $\chi^2_2 = exp(1/2)$, the exponential distribution with mean 2.

Extending Theorem 3

Theorem 7 *Suppose that, independently,* $X_j \sim \mathrm{N}(\alpha u_j, \sigma^2)$ *for* $j = 1, \ldots, n$, *where* $\mathbf{u} = (u_1, \ldots, u_n)^T$ *is a vector of unit length and* $\alpha \in \mathbf{R}$, *then*

$$\mathbf{u}^T\mathbf{X} = \sum_{j=1}^{n} u_j X_j$$

and

$$W^2 = \sum_{j=1}^{n} (X_j - u_j \mathbf{u}^T\mathbf{X})^2$$

are independent,

$$\mathbf{u}^T\mathbf{X} \sim \mathcal{N}(\alpha, \sigma^2)$$

$$\frac{W^2}{\sigma^2} \sim \chi^2_{n-1}$$

Proof.

Write $X_j = \alpha u_j + \sigma Z_j$, where $Z_j \sim \mathcal{N}(0,1)$, then

$$\mathbf{u}^T\mathbf{X} = \alpha + \sigma\mathbf{u}^T\mathbf{Z}$$

as $\sum u_j^2 = 1$, and

$$\begin{aligned} W^2 &= \sum_{j=1}^{n} (\alpha u_j + \sigma Z_j - u_j\alpha - u_j\sigma\mathbf{u}^T\mathbf{Z})^2 \\ &= \sum_{j=1}^{n} \sigma^2 (Z_j - u_j\mathbf{u}^T\mathbf{Z})^2 \end{aligned}$$

and so

$$\frac{W^2}{\sigma^2} = \sum_{j=1}^{n} (Z_j - u_j\mathbf{u}^T\mathbf{Z})^2$$

Let $\mathbf{Y} = Q\mathbf{Z}$, Q orthogonal, $\mathbf{u}^T$ as first row, then as before $Y_j, j = 1, \ldots, n$ are i.i.d. $\mathcal{N}(0,1)$,

$$Y_1 = \mathbf{u}^T\mathbf{Z}$$

and

$$
\begin{aligned}
&\sum_{j=1}^{n} (Z_j - u_j \mathbf{u}^T \mathbf{Z})^2 \\
&\quad = \sum_{j=1}^{n} Z_j^2 - (\mathbf{u}^T \mathbf{Z})^2 \\
&\quad = \sum_{j=1}^{n} Y_j^2 - Y_1^2 \\
&\quad = \sum_{j=2}^{n} Y_j^2 \\
&\quad \sim \chi^2_{n-1},
\end{aligned}
$$

independent of Y_1.

Similarly show

Theorem 8 *Suppose that, independently,* $X_j \sim \mathrm{N}(\alpha u_j + \beta v_j, \sigma^2)$ *for* $j = 1, \ldots, n$*, where* $\mathbf{u}, \mathbf{v}$ *are orthogonal vectors of unit length:* $\mathbf{u}^T\mathbf{v} = 0$*, and* $\alpha, \beta \in \mathbf{R}$*. Then*

$$\mathbf{u}^T\mathbf{X}, \mathbf{v}^T\mathbf{X},$$

and

$$W^2 = \sum_{j=1}^{n} (X_j - u_j\mathbf{u}^T\mathbf{X} - v_j\mathbf{v}^T\mathbf{X})^2$$

are independent,

$$\mathbf{u}^T\mathbf{X} \sim \mathcal{N}(\alpha, \sigma^2)$$

$$\mathbf{v}^T\mathbf{X} \sim \mathcal{N}(\beta, \sigma^2)$$

$$\frac{W^2}{\sigma^2} \sim \chi^2_{n-2}$$

Proof.

Write $X_j = \alpha u_j + \beta v_j + \sigma Z_j$, where $Z_j \sim \mathcal{N}(0,1)$, then

$$\mathbf{u}^T \mathbf{X} = \alpha + \sigma \mathbf{u}^T \mathbf{Z}$$

and

$$\mathbf{v}^T \mathbf{X} = \beta + \sigma \mathbf{v}^T \mathbf{Z}$$

and similarly to before

$$\frac{W^2}{\sigma^2} = \sum_{j=1}^{n} (Z_j - u_j \mathbf{u}^T \mathbf{Z} - v_j \mathbf{v}^T \mathbf{Z})^2$$

Let $\mathbf{Y} = Q\mathbf{Z}$, Q orthogonal, $\mathbf{u}^T$ as first row, $\mathbf{v}^T$ as second row, then as before $Y_j, j = 1, \ldots, n$ are i.i.d. $\mathcal{N}(0,1)$,

$$Y_1 = \mathbf{u}^T \mathbf{Z}$$

$$Y_2 = \mathbf{v}^T \mathbf{Z}$$

and

$$
\begin{aligned}
&\sum_{j=1}^n (Z_j - u_j \mathbf{u}^T \mathbf{Z} - v_j \mathbf{v}^T \mathbf{Z}))^2 \\
&\quad = \sum_{j=1}^n Z_j^2 - (\mathbf{u}^T \mathbf{Z})^2 - (\mathbf{v}^T \mathbf{Z})^2 \\
&\quad = \sum_{j=1}^n Y_j^2 - Y_1^2 - Y_2^2 \\
&\quad = \sum_{j=3}^n Y_j^2 \\
&\quad \sim \chi^2_{n-2},
\end{aligned}
$$

independent of Y_1, Y_2. Argueing as before finishes the proof.

Appendix: Transformation formula

Let $X_1, \ldots, X_n$ have joint probability density $f(x_1, \ldots, x_n)$, put

$$U_i = g_i(X_1, \ldots, X_n) \quad i = 1, \ldots, n$$

and assume that this gives a 1-1 transformation with inverse

$$X_i = h_i(U_1, \ldots, U_n) \quad i = 1, \ldots, n.$$

The *Jacobian* is

$$J = det \left[\left(\frac{\partial h_i}{\partial u_j} \right)_{i,j=1,\ldots,n} \right]$$

Then the joint density of $U_1, \ldots, U_n$ is

$$f(h_1(U_1, \ldots, U_n), \ldots, h_n(U_1, \ldots, U_n))|J|.$$