Abstracts

Cutting random recursive trees, and the Bolthausen–Sznitman coalescent

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(joint work with James Martin)

The Bolthausen–Sznitman coalescent was introduced in the context of spin glasses in [1]. These days, it is usually thought of as a special case of a more general class of coalescent processes introduced by Pitman [5] and Sagitov [7] and usually referred to as the Λ-*coalescents*. These are Markov processes taking values in the space $\mathcal{P}_\infty$ of partitions of $\mathbb{N}$, or the space $\mathcal{P}_n$ of partitions of $\{1, 2, \ldots, n\}$, where the blocks of the partition represent particles which gradually coalesce over the course of time. The dynamics of the Bolthausen–Sznitman coalescent on $\mathcal{P}_n$ are very simple. Suppose that we have an initial state $\pi \in \mathcal{P}_n$ consisting of b blocks. Then any k of them coalesce at rate

$$\lambda_{b,k} = \frac{(k-2)!(b-k)!}{(b-1)!}, \quad 2 \leq k \leq b \leq n,$$

regardless of block sizes or which integers the blocks contain. Since the state-space $\mathcal{P}_n$ is finite, the distribution of the coalescent is entirely specified by its initial distribution and these transition rates.

A random partition Π of $[n]$ is *exchangeable* if

$$\mathbb{P}(\Pi = \pi) = \mathbb{P}(\Pi = \sigma(\pi))$$

for any $\pi \in \mathcal{P}_n$ and any permutation σ of $[n]$. A random partition of $\mathbb{N}$ is exchangeable if its restriction to $[n]$ is exchangeable in the above sense for all $n \geq 1$. The above dynamics preserve the property of exchangeability: if the initial state of the Bolthausen–Sznitman coalescent is exchangeable, then the state remains exchangeable for all times. Moreover, the rates are such that the restriction of the coalescent evolving in $\mathcal{P}_{n+1}$ to $[n]$ evolves exactly as the coalescent evolving in $\mathcal{P}_n$; in other words, we have *consistency* for each $n \geq 1$. This means that we can define the coalescent evolving in $\mathcal{P}_\infty$ simply as a projective limit.

Let $(\Pi(t), t \geq 0)$ be the Bolthausen–Sznitman coalescent in $\mathcal{P}_\infty$. An important consequence of the exchangeability of $\Pi(t)$ for all $t \geq 0$ is that its blocks possess asymptotic frequencies i.e. if B is a block of $\Pi(t)$ then

$$\lim_{n \to \infty} \frac{|B \cap [n]|}{n}$$

exists almost surely.

We now turn to random recursive trees. A *recursive tree* is a labelled, unordered tree which is rooted at its vertex of smallest label and has the property that its labels increase along non-backtracking paths away from the root. We allow any partition of $[n]$ to be a label-set for such a tree, where we order blocks according to their least elements; the canonical label-set is the partition into singletons

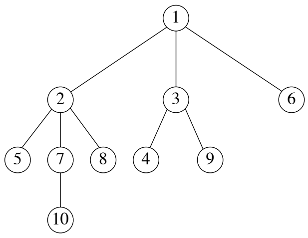

FIGURE 1. A recursive tree with label set [10].

$(\{1\}, \{2\}, \ldots, \{n\})$ for some $n \geq 1$ (see Figure 1). A *random recursive tree* on label-set $L = (\ell_1, \ell_2, \ldots, \ell_b)$ with $\ell_1 \preceq \ell_2 \preceq \ldots \preceq \ell_b$ is simply chosen uniformly at random from the $(b-1)!$ recursive trees with those labels. It is more easily constructed via a recursive procedure:

- start from a single vertex labelled by ℓ_1;
- for $k \geq 2$, attach a vertex labelled by ℓ_k to one of the vertices labelled by $\ell_1, \ldots, \ell_{k-1}$ chosen uniformly at random.

(Note that this procedure does not, in fact, require finiteness of the label-set.) We now consider a consider a variant of a cutting procedure which was first introduced by Meir and Moon [3, 4] and has been subsequently much studied in the combinatorics literature. Pick an edge uniformly at random. Cut it, and combine all of the labels below the cut edge with the label of the vertex just above. Repeat, until only the root remains (necessarily labelled by $[n]$). If we start with a partition of $[n]$ then, at every subsequent step, we clearly obtain a coarser partition of $[n]$. We can easily put this procedure into continuous time by associating an independent standard exponential random variable with each edge: this random variable gives the time at which that edge will be cut, if it still exists in the tree at that time.

Suppose that the tree has the partition of $[n]$ into singletons as its initial label-set. Let $\Gamma^{[n]}(t)$ be the partition obtained by running the cutting procedure for time t.

Theorem 1. *The process $(\Gamma^{[n]}(t), t \geq 0)$ is the Bolthausen–Sznitman coalescent on $[n]$.*

The proof is straightforward and relies on the fact that a random recursive tree cut at a uniformly-chosen edge is again a random recursive tree on its new label-set. Moreover, the rate at which a cut results in a coalescence of k labels is

$$\frac{(k-2)!(b-k)!}{(b-1)!}, \quad 2 \leq k \leq b \leq n.$$

We immediately recognise the rates of the Bolthausen–Sznitman coalescent. See [2] for the details of the proof.

Note that, because of the recursive way in which the tree is built, we have consistency in n and so, in fact, we can define $(\Gamma(t), t \geq 0)$ evolving in $\mathcal{P}_\infty$ by means of the cutting procedure applied to a random recursive tree labelled by $\mathbb{N}$.

The representation given by Theorem 1 is somewhat surprising. It splits the randomness of the coalescent into two parts: the randomness used to build the tree, and the randomness used to cut it. A particular realisation of the tree corresponds to a particular conditioning of the path of the coalescent. For example, if $\{2\}$ and $\{5\}$ are both children of $\{1\}$ then we condition 2 and 5 only to be in the same block once they have both coalesced with 1. The tree representation gives a size-biased viewpoint rather than the usual exchangeable one. The block containing 1 (which is always the label of the root) is a size-biased pick from amongst the blocks and so tends to be large. We can think of it as a tagged particle, and we watch the coalescent evolve from its point of view. This leads to a rather nice way to prove the following properties of the coalescent, originally due to Bolthausen and Sznitman [1] and Pitman [5] respectively. Write $\mathrm{PD}(\alpha, \theta)$ for the Poisson–Dirichlet distribution with parameters $0 < \alpha < 1$ and $\theta > -\alpha$ (see Pitman and Yor [6]).

Theorem 2. (1) *Write $F(t)$ for the asymptotic frequencies of the blocks of $(\Pi(t), t \geq 0)$, where the frequencies are listed in decreasing order of size. Then*

$$F(t) \sim \mathrm{PD}(e^{-t}, 0).$$

(2) *Write $F_*(t)$ for the frequency of the block containing 1 at time t. Then $(F_*(t), t \geq 0)$ is Markovian, with the same distribution as the process $(\gamma(1-e^{-t})/\gamma(1), t \geq 0)$, where $(\gamma(s), s \geq 0)$ is a Gamma subordinator. This entails that $F_*(t) \sim \mathrm{Beta}(1-e^{-t}, e^{-t})$. Moreover, if $J_1 \geq J_2 \geq \ldots \geq 0$ is the ranked sequence of jumps of $(F_*(t), t \geq 0)$ then $(J_1, J_2, \ldots) \sim \mathrm{PD}(0, 1)$.*

We refer the reader to [2] for the proofs and for further development.

Open problem. Find another exchangeable coalescent which may be represented by cutting down a combinatorial tree.

References

[1] E. Bolthausen and A.-S. Sznitman, *On Ruelle's probability cascades and an abstract cavity method*, Comm. Math. Phys. **197**(2) (1998), 247–276.

[2] C. Goldschmidt and J.B. Martin, *Random recursive trees and the Bolthausen–Sznitman coalescent*, Electron. J. Probab. **10** (2005), Paper no. 21, 718–745.

[3] A. Meir and J. W. Moon. *Cutting down random trees*, J. Austral. Math. Soc. **11** (1970), 313–324.

[4] A. Meir and J. W. Moon. *Cutting down recursive trees*, Math. Bioscience **21** (1974), 173–181.

[5] J. Pitman, *Coalescents with multiple collisions*, Ann. Probab. **27**(4) (1999), 1870–1902.

[6] J. Pitman and M. Yor, *The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator*, Ann. Probab. **25** (1997), 855–900.

[7] S. Sagitov, *The general coalescent with asynchronous mergers of ancestral lines*, J. Appl. Probab. **36**(4) (1999), 1116–1125.

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