Warwick, 27th April 2009

Lecture 3: The scaling limit of critical random graphs (continued)

and then

An introduction to fragmentation processes

Christina Goldschmidt

Recap from last week

We consider the Erdős-Rényi random graph $G(n, p)$ with p in the critical window, i.e. $p = 1/n + \lambda n^{-4/3}$. In this parameter range, the largest components are of size $\Theta(n^{2/3})$.

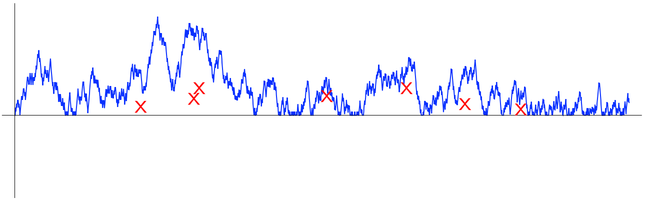
Aldous (1997) describes the limit of the sequence of component sizes and surpluses:

$$(n^{-2/3}(C_1^n, C_2^n, \ldots), (S_1^n, S_2^n, \ldots)) \xrightarrow{d} ((C_1, C_2, \ldots), (S_1, S_2, \ldots))$$

as $n \to \infty$. These limits are obtained from

$$W^\lambda(t) = W(t) + \lambda t - t^2/2,$$

a Brownian motion with parabolic drift, which is then reflected at its minimum, and a rate one Poisson point process in the plane.



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In the limit, these spanning trees converge to CRT's coded by the excursions of Aldous' limit process. We call these tilted excursions and tilted trees. The surplus edges become vertex-identifications, whose locations are coded by the locations of the Poisson points under the excursion.

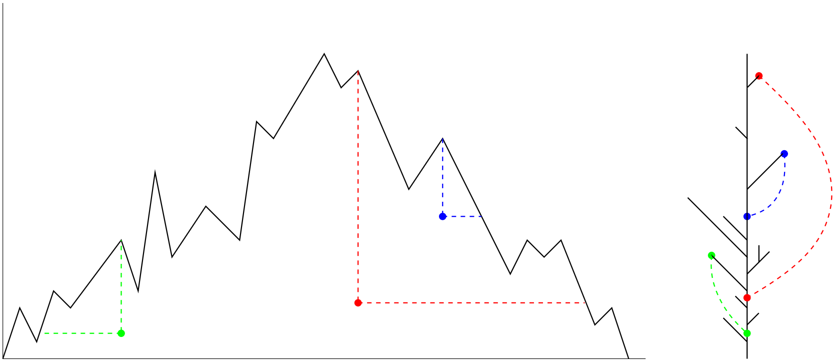
Tilted excursions

An excursion $\tilde{e}^{(x)}$ of Aldous' limit process, conditioned to have length x, has a distribution specified by

$$\mathbb{E}\left[f\left(\tilde{e}^{(x)}\right)\right] = \frac{\mathbb{E}\left[f\left(e^{(x)}\right)\exp\left(\int_0^x e^{(x)}(u)du\right)\right]}{\mathbb{E}\left[\exp\left(\int_0^x e^{(x)}(u)du\right)\right]},$$

where f is any suitable test-function and $e^{(x)}$ is a Brownian excursion of length x.

Vertex identifications



A point at (x, y) identifies the vertex v at height $h(x)$ with the vertex at distance y along the path from the root to v.

Convergence result

Let $\mathcal{C}_1^n, \mathcal{C}_2^n, \ldots$ be the sequence of components of $G(n,p)$ in decreasing order of size, considered as metric spaces with the graph distance.

Theorem. As $n \to \infty$,

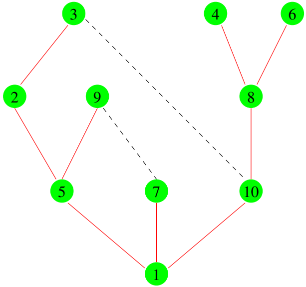
$$n^{-1/3}(\mathcal{C}_1^n, \mathcal{C}_2^n, \ldots) \xrightarrow{d} (\mathcal{C}_1, \mathcal{C}_2, \ldots),$$

where $\mathcal{C}_1, \mathcal{C}_2, \ldots$ is the sequence of metric spaces corresponding to the excursions of Aldous' marked limit process in decreasing order of length.

Here, convergence is with respect to the metric

$$d(\mathcal{A}, \mathcal{B}) := \left(\sum_{i=1}^{\infty} d_{GH}(\mathcal{A}_i, \mathcal{B}_i)^4 \right)^{1/4}.$$

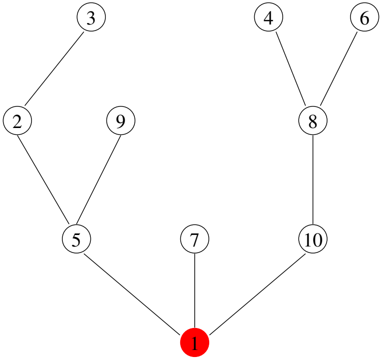
Via a depth-first exploration of a component, we define the depth-first tree.



The edges which make no difference to the depth-first exploration are called permitted.

Depth-first exploration

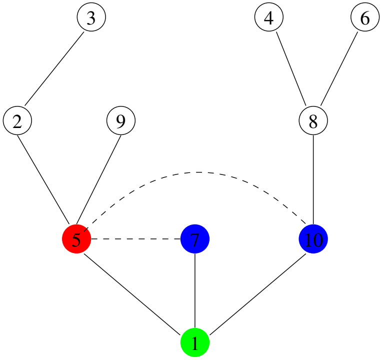
Step 0: initialization



Current: 1 Alive: none Dead: none $X(0) = 0$.

Depth-first exploration

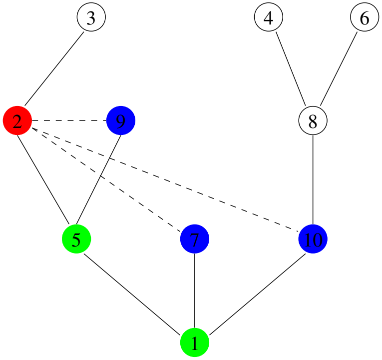
Step 1



Current: 5 Alive: 7, 10 Dead: 1 $X(1) = 2.$

Depth-first exploration

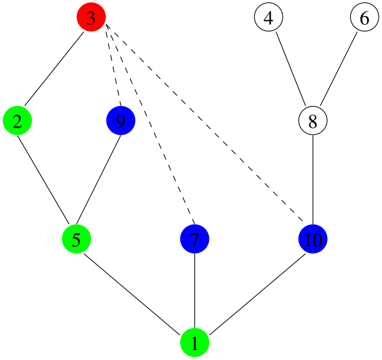
Step 2



Current: 2 Alive: 9, 7, 10 Dead: 1, 5 $X(2) = 3$.

Depth-first exploration

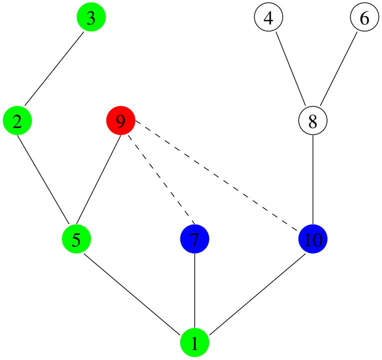
Step 3



Current: 3 Alive: 9, 7, 10 Dead: 1, 5, 2 $X(3) = 3$.

Depth-first exploration

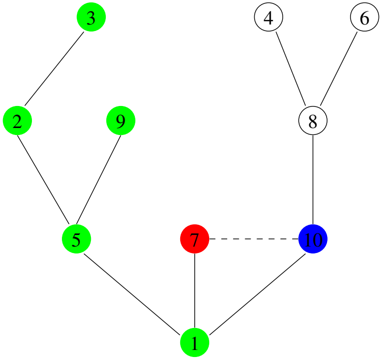
Step 4



Current: 9   Alive: 7, 10   Dead: 1, 5, 2, 3 $X(4) = 2$.

Depth-first exploration

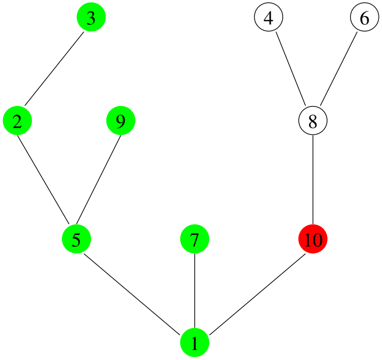
Step 5



Current: 7   Alive: 10   Dead: 1, 5, 2, 3, 9 $X(5) = 1$.

Depth-first exploration

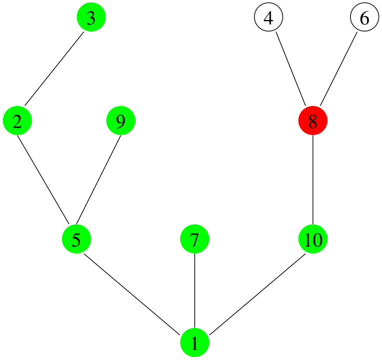
Step 6



Current: 10 Alive: none Dead: 1, 5, 2, 3, 9, 7 $X(6) = 0$.

Depth-first exploration

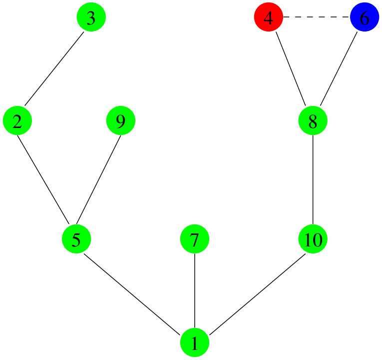
Step 7



Current: 8 Alive: none Dead: 1, 5, 2, 3, 9, 7, 10 $X(7) = 0$.

Depth-first exploration

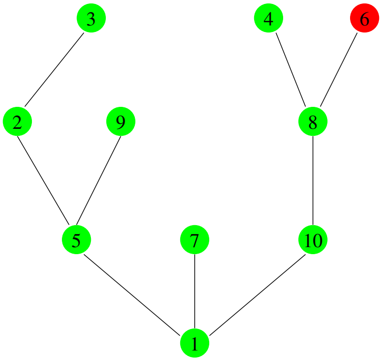
Step 8



Current: 4   Alive: 6   Dead: 1, 5, 2, 3, 9, 7, 10, 8 $X(8) = 1$.

Depth-first exploration

Step 9



Current: 6 Alive: none Dead: 1, 5, 2, 3, 9, 7, 10, 8, 4

$X(9) = 0$.

The number of permitted edges for a labelled tree T is

$$a(T) := \sum_{k=0}^{m-1} X(k),$$

where $(X(k), 0 \leq k \leq m-1)$ is the depth-first walk of T.

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The set of connected graphs with label-set $[m] := \{1, 2, \ldots, m\}$ can be partitioned by depth-first tree.

Recipe for creating a connected graph on $[m]$

Create a connected graph $\tilde{G}_m^p$ as follows.

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- Add each of the $a(\tilde{T}_m^p)$ permitted edges to $\tilde{T}_m^p$ independently with probability p.

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Proof. For a connected graph G on $[m]$ which has $T(G) = T$ and surplus s,

$$\mathbb{P}\left(\tilde{G}_m^p = G\right) \propto (1-p)^{-a(T)} p^s (1-p)^{a(T)-s} = (p/(1-p))^s.$$

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Likewise, by the definition of $G(n,p)$,

$$\begin{aligned} \mathbb{P}(G_m^p = G) &\propto \mathbb{P}(G(m,p) = G) \\ &= p^{m+s-1}(1-p)^{\binom{m}{2}-(m+s-1)} \propto (p/(1-p))^s. \end{aligned}$$

□

Taking limits

So we need to prove that

- the tree $\tilde{T}_m^p$ converges to a CRT coded by a tilted excursion;
- the locations of the surplus edges converge to the locations in our limiting picture.

We will deal with the tree first. For simplicity, we will take $p = m^{-3/2}$; the general case is similar.

Part 1: Convergence of the tree

Write $\tilde{X}^m$ for the depth-first walk associated with $\tilde{T}^p_m$, thought of as a càdlàg function $[0, m] \to \mathbb{R}^+$. Then

$$a\left(\tilde{T}^p_m\right) = \int_0^m \tilde{X}^m(u)du.$$

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Recall that T_m is a uniform random tree on $[m]$ and that X^m is its depth-first walk. Then

$$(m^{-1/2}X^m(mt), 0 \le t \le 1) \xrightarrow{d} (e(t), 0 \le t \le 1).$$

Part 1: Convergence of the tree

Now use the change of measure to get from $\tilde{X}^m$ to X^m: for any bounded continuous function f,

$$\mathbb{E}\left[f\left(m^{-1/2}\tilde{X}^m(mt), 0 \le t \le 1\right)\right] = \frac{\mathbb{E}\left[f\left(m^{-1/2}X^m(mt), 0 \le t \le 1\right)(1-p)^{-\int_0^m X^m(u)du}\right]}{\mathbb{E}\left[(1-p)^{-\int_0^m X^m(u)du}\right]}$$

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$$= \frac{\mathbb{E}\left[f\left(m^{-1/2}X^m(mt), 0 \le t \le 1\right)(1-p)^{-m^{3/2}\int_0^1 m^{-1/2}X^m(ms)ds}\right]}{\mathbb{E}\left[(1-p)^{-m^{3/2}\int_0^1 m^{-1/2}X^m(ms)ds}\right]}$$

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Since $(m^{-1/2}X^m(mt), 0 \le t \le 1) \xrightarrow{d} (e(t), 0 \le t \le 1)$ and $p = m^{-3/2}$,

$$(1-p)^{-m^{3/2}\int_0^1 m^{-1/2}X^m(ms)ds} \xrightarrow{d} \exp\left(\int_0^1 e(u)du\right).$$

Part 1: Convergence of the tree

Taking care with the limits, we obtain

$$\mathbb{E}\left[f\left(m^{-1/2}\tilde{X}^m(mt), 0 \le t \le 1\right)\right] \to \frac{\mathbb{E}\left[f(e)\exp\left(\int_0^1 e(u)du\right)\right]}{\mathbb{E}\left[\exp\left(\int_0^1 e(u)du\right)\right]}$$

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As in the uniform case, we get jointly

$$(m^{-1/2}\tilde{H}^m(mt), 0 \le t \le 1) \xrightarrow{d} (2\tilde{e}(t), 0 \le t \le 1)$$

$$(m^{-1/2}\tilde{C}^m(2mt), 0 \le t \le 1) \xrightarrow{d} (2\tilde{e}(t), 0 \le t \le 1)$$

(where the limit is the *same* tilted excursion).

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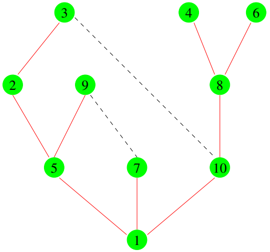
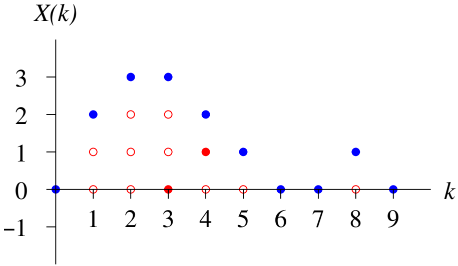
This entails that

$$\frac{1}{\sqrt{m}}\tilde{T}_m^p \xrightarrow{d} \tilde{\mathcal{T}}.$$

Part 2: Surplus edges

The permitted edges are in bijective correspondence with the integer points under the graph of the depth-first walk.





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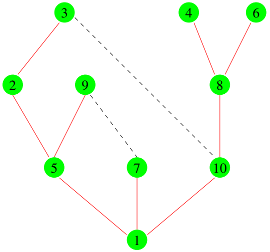
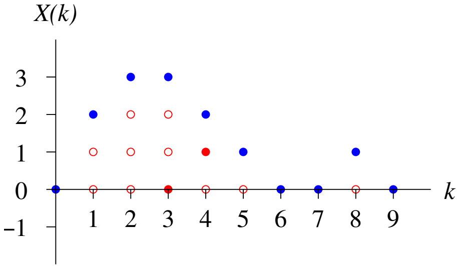
The permitted edges are in bijective correspondence with the integer points under the graph of the depth-first walk. Since each permitted edge is included independently with probability p, the surplus edges form a Binomial point process.





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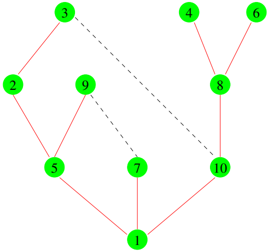
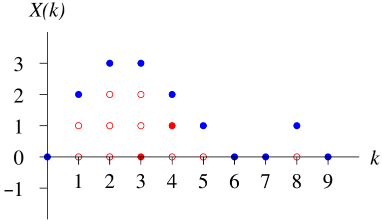
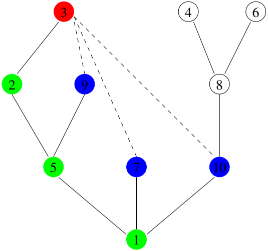
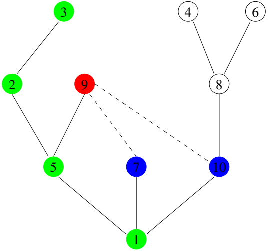




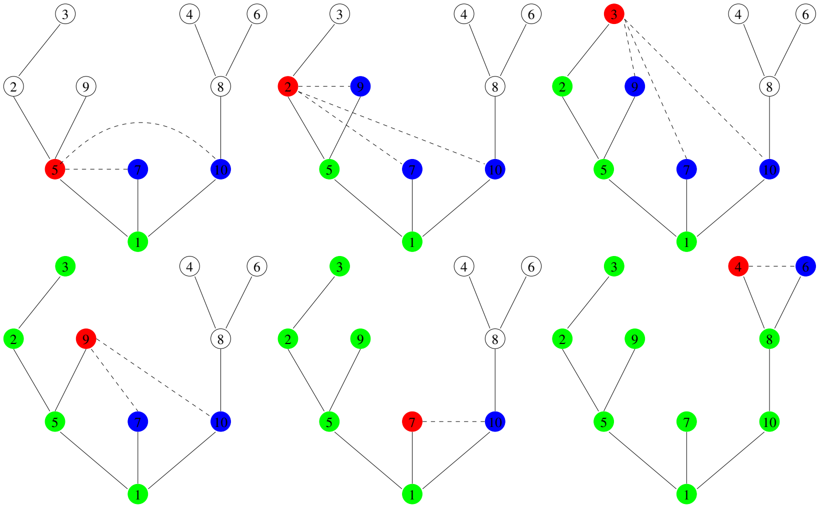




A point at (k, j) means "put an edge between the current vertex at step k and the vertex at distance j from the bottom of the list of alive vertices".

Part 2: Surplus edges

Surplus edges almost go to ancestors... In fact, they go to younger children of ancestors of the current vertex.



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When we rescale, the distance between a vertex and one of its children vanishes and so, in the limit, surplus "edges" do go to ancestors of the current vertex.

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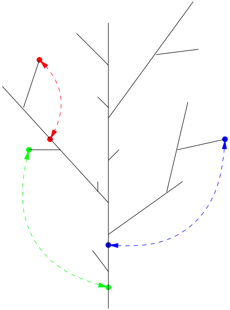
The Binomial point process of surplus edges, when rescaled, straightforwardly converges to the required Poisson point process.

The difference between the depth-first walk and the height process is also small, and so the locations of the surplus "edges" are essentially as described in our limit process.

□

References



L. Addario-Berry, N. Broutin and C. Goldschmidt,
The continuum limit of critical random graphs,
`arXiv:0903.4730 [math.PR]`.

L. Addario-Berry, N. Broutin and C. Goldschmidt
Distributional limits in critical random graphs,
in preparation.

An introduction to fragmentation processes

Fragmentation

We will talk about mathematical models for an object which splits apart randomly and repeatedly over the course of time.

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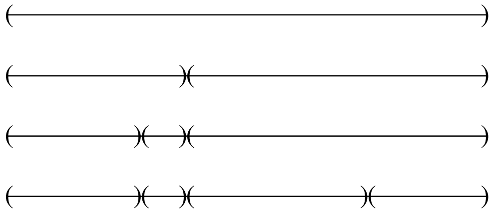
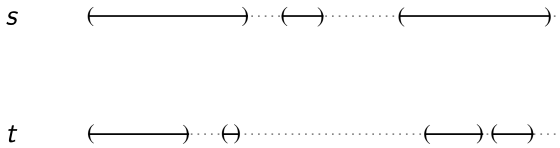
In this lecture, the object will essentially be the open interval $(0, 1)$, so that the successive states of a fragmentation process might look like this:



Definition. An interval fragmentation is a process $(O(t), t \geq 0)$ taking values in the set of open subsets of $(0, 1)$ such that $O(t) \subseteq O(s)$ whenever $0 \leq s \leq t$.



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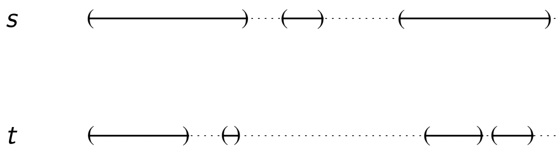

We will refer to the interval components of $O(t)$ as blocks.

Interval fragmentations derived from excursions

By an excursion, we will now mean a continuous function $f : [0, 1] \to \mathbb{R}^+$ such that $f(0) = f(1) = 0$ and $f(x) > 0$ for all $x \in (0, 1)$.

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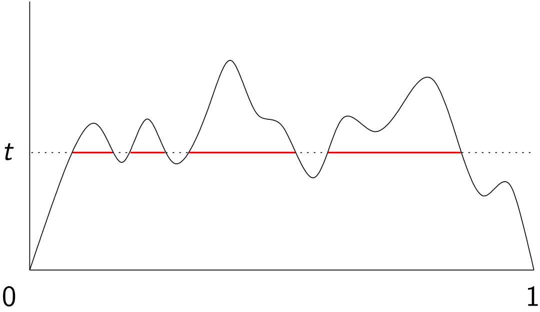
By an excursion, we will now mean a continuous function $f : [0,1] \to \mathbb{R}^+$ such that $f(0) = f(1) = 0$ and $f(x) > 0$ for all $x \in (0,1)$.

The associated interval fragmentation is given by

$$O(t) := \{x \in [0,1] : f(x) > t\}.$$



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- Markov property
- Different blocks split independently ("branching property")
- Self-similarity: the way in which different blocks split is the same each time.

It turns out to be easier to think in terms of the lengths of the blocks. By a ranked fragmentation, we mean an ordered list of the lengths of the blocks of $O(t)$, written $(F(t), t \geq 0)$.

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Here, $F(t)$ takes values in the set

$$\mathcal{S}^{\downarrow} = \left\{ \mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\}.$$

Self-similar fragmentations

The self-similar fragmentations are a particularly nice family of fragmentation processes, which were introduced by Jean Bertoin.

Self-similar fragmentations

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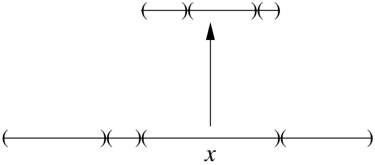
- a block of length x splits at a rate proportional to x^{α}, for some index $\alpha \in \mathbb{R}$, independently for different blocks;
- the relative lengths of the sub-blocks produced have the same distribution for each split.

At rate x^{α}:



Definition. A ranked self-similar fragmentation $(F(t), t \geq 0)$ with index $\alpha \in \mathbb{R}$ is a càdlàg Markov process taking values in $\mathcal{S}^{\downarrow}$ such that

- $F(0) = (1, 0, \ldots)$;
- conditional on $F(t) = (x_1, x_2, \ldots)$, $F(t+s)$ has the distribution of the decreasing rearrangement of the terms of

$$x_i F^{(i)}(x_i^{\alpha} s), \quad i \geq 1,$$

where $F^{(1)}, F^{(2)}, \ldots$ are i.i.d. copies of the original process F.

Note that this definition does not exclude the possibility that jumps of the process might be dense.

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Of course, there might be several different interval fragmentations corresponding to a particular ranked fragmentation. It is possible that not all, or indeed none, of them can be constructed starting from an excursion. We will come back to this point later.

An example

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This gives a self-similar fragmentation with index $\alpha = 1$.

Characterization

Theorem. (Bertoin (2002)) A ranked self-similar fragmentation is characterized by three parameters: (α, ν, c), $\alpha \in \mathbb{R}$, ν is a measure on $\mathcal{S}^{\downarrow}$ and $c \geq 0$.

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ν is the dislocation measure (satisfying $\int_{\mathcal{S}^{\downarrow}} (1 - s_1)\nu(d\mathbf{s}) < \infty$)
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If ν is finite, when we have a block of size m, it splits at rate $m^{\alpha}\nu(\mathcal{S}^{\downarrow})$ into blocks of sizes $(ms_1, ms_2, \ldots)$, where $\mathbf{s} = (s_1, s_2, \ldots)$ is sampled according to the distribution $\nu(\cdot)/\nu(\mathcal{S}^{\downarrow})$.

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The erosion coefficient describes a continuous melting of the blocks.

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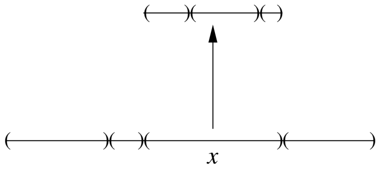
For the rest of this talk, we will have $c = 0$.

Behaviour

At rate x^{α}:



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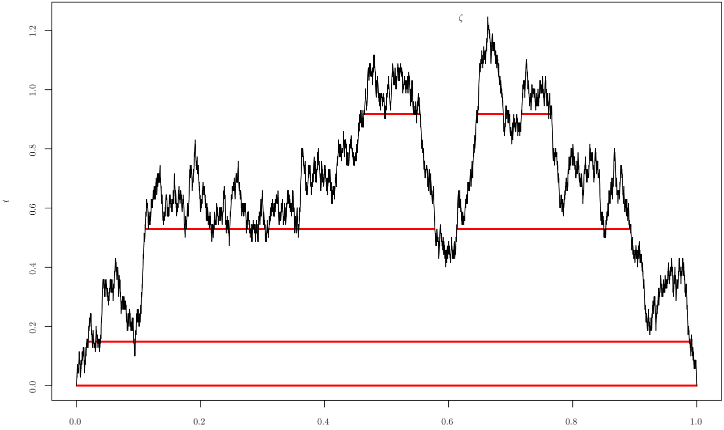
- If $\alpha > 0$, bigger blocks split faster. This tends to act to homogenize the block sizes. In fact, the empirical measure of the rescaled block sizes has law of large numbers-style behaviour as $t \to \infty$.
- If $\alpha < 0$, smaller blocks split faster than larger ones. In fact, as blocks get smaller, they split faster and faster until they are reduced to dust i.e. blocks of infinitesimal mass. The whole state is reduced to dust in an almost surely finite time ζ, called the extinction time.

An important example: the Brownian fragmentation

Take a standard Brownian excursion $(e(x), 0 \leq x \leq 1)$ and consider the associated interval fragmentation $(O(t), t \geq 0)$.



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Let $(F(t), t \geq 0)$ be the ranked fragmentation derived from the Brownian interval fragmentation $(O(t), t \geq 0)$.

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Theorem. (Bertoin (2002)) $(F(t), t \geq 0)$ is a self-similar fragmentation of index $\alpha = -1/2$ and binary dislocation measure specified by $\nu(s_1 + s_2 < 1) = 0$ and

$$\nu(s_1 \in \mathrm{d}x) = \frac{2}{\sqrt{2\pi x^3(1-x)^3}} \mathbb{I}_{[1/2,1]}(x)\mathrm{d}x.$$