# Scaling limits of random graphs: exercises

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1. **(Aldous-Broder algorithm)** The purpose of this exercise is to convince you that the Aldous-Broder algorithm generates a uniform random tree. We saw in lectures that the tree it generates is stationary, so we just need to prove that the stationary distribution is uniform. Recall that $\mathbb{T}_n^\bullet$ denotes the set of rooted trees labelled by $\{1, 2, \ldots, n\}$. Consider the *time-reversed chain* (which must have the same stationary distribution). For $\tau, \tau' \in \mathbb{T}_n^\bullet$, write $q(\tau, \tau')$ for the transition probability from $\tau$ to $\tau'$ for the time-reversed chain.

   (a) Argue that the chain is irreducible on $\mathbb{T}_n^\bullet$.

   (b) Show that for fixed $\tau$, $q(\tau, \tau') = 0$ or $1/(n-1)$.

   (c) Show that for fixed $\tau'$, $q(\tau, \tau') = 0$ or $1/(n-1)$.

   (d) It follows that $Q = (q(\tau, \tau'))_{\tau, \tau' \in \mathbb{T}_n^\bullet}$ is a doubly stochastic matrix. Deduce that the stationary distribution must be uniform.

2. **(Uniform random trees)** Let $T$ be a Galton-Watson tree with Poisson(1) offspring distribution and total progeny $N$.

   (a) Fix a particular rooted ordered tree $\mathrm{t}$ with $n$ vertices having numbers of children $c_v, v \in \mathrm{t}$. What is $\mathbb{P}(T = \mathrm{t})$?

   (b) Condition on the event $\{N = n\}$. Assign the vertices of $T$ a uniformly random labelling by $[n]$, and let $\tilde{T}$ be the labelled tree obtained by forgetting the ordering and the root. Show that $\tilde{T}$ has the same distribution as $T_n$, a uniform random tree on $n$ vertices.
   *Hint: it suffices to show that the probability of obtaining a particular tree* $\mathrm{t}$ *is a function of* $n$ *only.*

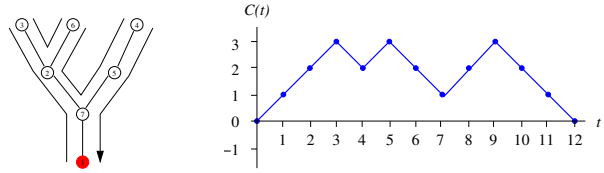
3. **(Other combinatorial trees)** Let $T$ be a Galton-Watson tree with offspring distribution $(p_k)_{k \geq 0}$ and total progeny $N$.

   (a) Show that if $p_k = 2^{-k-1}, k \geq 0$ then, conditional on $N = n$, $T$ is uniform on the set of ordered rooted trees with $n$ vertices.

   (b) Show that if $p_0 = 1/2$ and $p_2 = 1/2$ then, conditional on $N = n$ (for $n$ odd), $T$ is uniform on the set of complete binary trees.

4. **(Contour function and convergence to the Brownian CRT)** In Lecture 1, we discussed the depth-first walk and the height function of a tree. A third encoding which is often used is the so-called *contour function* $(C(i), 0 \leq i \leq 2(n-1))$. For a rooted ordered tree $\mathbf{t}$, we imagine a particle tracing the outline of the tree from left to right at speed 1. (The picture

below is for a labelled tree, with a planar embedding given by the labels.) Notice that we visit every vertex a number of times given by its degree.




Let $T_n$ be a Galton-Watson tree with offspring distribution $p(k) = \left(\frac{1}{2}\right)^{k+1}, k \geq 0$, conditioned to have total progeny $N = n$, as in Question 3(a). Let $(C^n(i), 0 \leq i \leq 2(n-1))$ be its contour function. It will be convenient to define a somewhat shifted version: let $\tilde{C}^n(0) = 0$, $\tilde{C}^n(2n) = 0$ and, for $1 \leq i \leq 2n-1$, $\tilde{C}^n(i) = 1 + C(i-1)$.

(a) Show that $(\tilde{C}^n(i), 0 \leq i \leq 2n))$ has the same distribution as a simple symmetric random walk (i.e. a random walk which makes steps of $+1$ with probability $1/2$ and steps of $-1$ with probability $1/2$) conditioned to return to the origin for the first time at time $2n$.
*Hint: first consider the unconditioned Galton-Watson tree with this offspring distribution.*

(b) It's straightforward to interpolate linearly to get a continuous function $\tilde{C}^n : [0, 2n] \to \mathbb{R}_+$. Let $\tilde{T}^n$ be the $\mathbb{R}$-tree encoded by this linear interpolation. Show that

$$\mathrm{d}_{\mathrm{GH}}(T_n, \tilde{T}_n) \leq \frac{1}{2}.$$

*Hint: notice that $T_n$ considered as a metric space has only $n$ points, whereas $\tilde{T}_n$ is an $\mathbb{R}$-tree and consists of uncountably many points. Draw a picture and find a correspondence.*

(c) Suppose that we have continuous excursions $f : [0,1] \to \mathbb{R}_+$ and $g : [0,1] \to \mathbb{R}_+$ which encode $\mathbb{R}$-trees $\mathcal{T}_f$ and $\mathcal{T}_g$. For $t \in [0,1]$, let $p_f(t)$ be the image of $t$ in the tree $\mathcal{T}_f$ and similarly for $p_g(t)$. Now define a correspondence

$$R = \left\{(x,y) \in \mathcal{T}_f \times \mathcal{T}_g : x = p_f(t),\ y = p_g(t) \text{ for some } t \in [0,1]\right\}.$$

Show that $\mathrm{dis}(R) \leq 4\|f - g\|_\infty$.
*Hint: recall how the metric in an $\mathbb{R}$-tree is related to the function encoding it.*

(d) Observe that the variance of the step-size in a simple symmetric random walk is 1. Hence, by Kaigh's theorem, we have

$$\frac{1}{\sqrt{2(n-1)}}(C^n(2(n-1)t), 0 \leq t \leq 1) \xrightarrow{d} (e(t), 0 \leq t \leq 1) \qquad (*)$$

as $n \to \infty$. Use this, (b) and (c) to prove directly that $\frac{1}{\sqrt{n}}T_n$ converges to a constant multiple of the Brownian CRT in the Gromov-Hausdorff sense.
*Hint: you may want to use Skorokhod's representation theorem in order to work on a probability space where the convergence (*) occurs almost surely.*

*A beautiful exposition of this approach is given by* Jean-François Le Gall & Grégory Miermont, **Scaling limits of random trees and planar maps**, Lecture notes for the Clay Mathematical Institute Summer School in Buzios, July 11 to August 7, 2010, available at `http://perso.ens-lyon.fr/gregory.miermont/Cours_Buzios.pdf`.

5. **(Generating a component of the Erdős–Rényi random graph conditionally on its size)** Let $\tilde{G}_m^p$ be a connected graph on vertices labelled by $[m]$ generated as follows:

   - Pick a random tree $\tilde{T}_m^p$ such that
   $$\mathbb{P}\left(\tilde{T}_m^p = T\right) \propto (1-p)^{-a(T)}, \quad T \in \mathbb{T}.$$
   - Add each of the $a(\tilde{T}_m^p)$ permitted edges independently with probability $p$.

   Show that $\tilde{G}_m^p$ has the same distribution as a component of $G(n,p)$ conditioned to have vertex set $[m]$.

6. **(Configuration model)** Fix a degree sequence $d_1, \ldots, d_n$. Show that the probability of generating a particular multigraph $G$ with these degrees is
$$\frac{1}{(\ell_n - 1)!!} \frac{\prod_{i=1}^n d_i!}{2^{\mathrm{sl}(G)} \prod_{e \in E(G)} \mathrm{mult}(e)!},$$
   where $\ell_n = \sum_{i=1}^n d_i$, $\mathrm{sl}(G)$ is the number of self-loops in $G$ and $\mathrm{mult}(e)$ is the multiplicity of the edge $e \in E(G)$.

7. **(Size-biased random variables)** Suppose that $X$ is a non-negative integer-valued random variable with finite mean, and let $X^*$ have the *size-biased distribution*
$$\mathbb{P}\left(X^* = k\right) = \frac{k\mathbb{P}\left(X = k\right)}{\mathbb{E}\left[X\right]}, \quad k \geq 1.$$
   Show that $X^* \stackrel{d}{=} 1 + X$ if and only if $X$ has a Poisson distribution.

8. **(Size-biased order and absolute continuity)** Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with finite mean, and let $(\hat{X}_1^n, \hat{X}_2^n, \ldots, \hat{X}_n^n)$ be the same random variables in size-biased random order. Let $X_1^*, X_2^*, \ldots$ be i.i.d. with the (true) size-biased distribution. Show that for $m < n$,
$$\begin{aligned}&\mathbb{P}\left(\hat{X}_1^n = k_1, \hat{X}_2^n = k_2, \ldots, \hat{X}_m^n = k_m\right)\\ &\quad = \frac{n!\mu^m}{(n-m)!}\mathbb{E}\left[\prod_{i=1}^m \frac{1}{\sum_{j=i}^m k_j + \Xi_{n-m}}\right]\mathbb{P}\left(X_1^* = k_1, X_2^* = k_2, \ldots, X_m^* = k_m\right),\end{aligned}$$
   where $\Xi_{n-m} \stackrel{d}{=} X_1 + \ldots + X_{n-m}$.