Advanced Simulation

Problem Sheet 1

Exercise 1

1. Let $Y \sim \mathcal{E}xp(\lambda)$ and let $a > 0$. We consider the variable after restricting its support to be $[a, +\infty)$. That is, let $X = Y|Y \geq a$, i.e. X has the law of Y conditionally on being in $[a, +\infty)$. Calculate $F_X(x)$, the cumulative distribution function of X, and $F_X^{-1}(u)$, the quantile function of X. Describe an algorithm to simulate X from $U \sim \mathcal{U}_{[0,1]}$.

2. Let a and b be given, with $a < b$. Show that we can simulate $X = Y|a \leq Y \leq b$ from $U \sim \mathcal{U}_{[0,1]}$ using
$$X = F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U),$$
i.e. show that if X is given by the formula above, then $\Pr(X \leq x) = \Pr(Y \leq x|a \leq Y \leq b)$. Apply the formula to simulate an exponential random variable conditioned to be greater than a, as in the previous question.

3. Here is a simple algorithm to simulate $X = Y|Y > a$ for $Y \sim \mathcal{E}xp(\lambda)$:

    (a) Let $Y \sim \mathcal{E}xp(\lambda)$. Simulate $Y = y$.

    (b) If $Y > a$ then stop and return $X = y$, and otherwise, start again at step (a).

    Show that this is just a rejection algorithm, by writing the proposal and target densities π and q, as well as the bound $M = \max_x \pi(x)/q(x)$. Calculate the expected number of trials to the first acceptance. Why is inversion to be preferred for $a \gg 1/\lambda$?

Exercise 2

1. Let X_1, X_2 be two independent random variables with $X_1 \sim \mathcal{G}amma\,(a, 1)$ and $X_2 \sim \mathcal{G}amma\,(b, 1)$. Show that $R = X_1/\left(X_1 + X_2\right)$ and $S = X_1 + X_2$ are independent and that $R \sim \mathcal{B}eta\,(a, b)$, $S \sim \mathcal{G}amma\,(a + b, 1)$. Recall that Gamma and Beta densities are
$$f_\Gamma\left(x; \alpha, \beta\right) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma\left(\alpha\right)},\ x \in (0, \infty)$$
and
$$f_B\left(x; a, b\right) = \frac{\Gamma\left(a+b\right)}{\Gamma\left(a\right)\Gamma\left(b\right)} x^{a-1}\left(1-x\right)^{b-1},\ x \in \left(0, 1\right).$$

2. Let $U \sim \mathcal{U}_{[0,1]}$ and $a > 0$. Show that $X = U^{1/a} \sim \mathcal{B}eta\,(a, 1)$.

3. Let $U \sim \mathcal{U}_{[0,1]}$ and $V \sim \mathcal{U}_{[0,1]}$ be independent and $a \in (0, 1)$. For $Y = U^{1/a}$ and $Z = V^{1/(1-a)}$, calculate
$$\mathbb{P}\left(\frac{Y}{Y+Z} \leq t, Y + Z \leq 1\right)$$
for any $t \in (0, 1)$ and deduce that the conditional distribution of $W = Y/\left(Y+Z\right)$ given $Y + Z \leq 1$ is $\mathcal{B}eta\,(a, 1 - a)$. *(Hint: writing both inequalities as constraints on Z could ease the calculation).*

4. In the setting of (3), show that the conditional distribution of TW given $Y + Z \leq 1$, for an independent $T \sim \mathcal{E}xp\,(1) = \mathcal{G}amma\,(1, 1)$ random variable, is $\mathcal{G}amma\,(a, 1)$.

5. Let $a \in (0,1)$.

   (a) Simulate two independent $U, V \sim \mathcal{U}_{[0,1]}$.
   (b) Set $Y = U^{1/a}$ and $Z = V^{1/(1-a)}$.
   (c) If $(Y + Z) \leq 1$ go to (d), else go to (a).
   (d) Simulate an independent $A \sim \mathcal{U}_{[0,1]}$ and set $T = -\log(A)$.
   (e) Return $TY/(Y+Z)$.

   What is this procedure doing? Explain its relevance for simulations.

6. Based on (1), explain how you can generate a $\mathcal{B}eta\,(a,b)$ random variable from a sequence of $\mathcal{U}_{[0,1]}$ random variables, for any $a > 0$ and $b > 0$. *(Hint: consider $a \in (0,1)$ first and use the additivity of Gamma variables to generate $\mathcal{G}amma\,(a,1)$ variables, from which the Beta variables can be constructed).*

Exercise 3

The R questions are optional and should not be handed back. The solution will not be covered in the classes, but will be directly posted on the course's website.

1. Reproduce the figures on the estimation of the number π in the lecture.