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Self–similar Energy Forms on the Sierpinski Gasket with Twists

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Abstract By introducing twists into the iterated function system that defines the Sierpinski gasket, we are able to construct a unique regular energy form that satisfies a self–similar identity with any prescribed projective weights. Our construction is explicit (involving finding a root of a 4th order polynomial), and we are able to find explicitly a polynomial identity for the algebraic variety containing the smooth manifold of admissible weights. Without the twists, there are obstructions to existence, and a complete description due to Sabot is quite complicated.

Keywords Analysis on fractals · Self-similar energy forms · Sierpinski gasket

Mathematics Subject Classifications (2000) 28A80 · 31C45

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1 Introduction

One approach to analysis on fractals, as developed by Kigami [4–7] (see [19, 21] for expository accounts), is to construct an energy form, analogous to

$$\mathcal{E}(u) = \int_0^1 |u'(x)|^2 dx \tag{1}$$

on the unit interval. To be specific, suppose the fractal is given as the invariant set K of an iterated function system (IFS) of contractive similarities $\{F_i\}$ on some Euclidean space, so

$$K = \bigcup_i F_i K. \tag{2}$$

We seek a strongly local regular Dirichlet form $\mathcal{E}$ on a domain (dom $\mathcal{E}$) dense in the continuous functions, that satisfies a self–similar identity

$$\mathcal{E}(u) = \sum_i r_i^{-1} \mathcal{E}(u \circ F_i) \tag{3}$$

for a set of weights $\{r_i\}$ satisfying

$$0 < r_i < 1. \tag{4}$$

We restrict attention to a class of fractals, called *postcritically finite*, with the following properties: K is connected and contains a finite set V_0 (called the *boundary* of K) such that

$$F_i K \cap F_j K \subseteq F_i V_0 \cap F_j V_0. \tag{5}$$

Then we approximate K from within by a sequence of graphs $\{\Gamma_m\}$ with vertices $\{V_m\}$ and edge relation $x \underset{m}{\sim} y$ as follows: Γ_0 is the complete graph on V_0, and

$$V_m = \bigcup_{i=1}^{m} V_{m-1} = \bigcup_{|w|=m} F_w V_0,$$

where $w = (w_1, \ldots, w_m)$ is a word of length $|w| = m$, and $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$. We define $x \underset{m}{\sim} y$ if and only if there exists w of length m with $x, y \in F_w V_0$.

The existence of a self–similar energy form is equivalent to the existence of a solution to the following renormalization problem. Consider the space of all discrete energy forms on Γ_0,

$$\mathcal{E}_0(u) = \sum_{x \underset{0}{\sim} y} c(x, y)(u(x) - u(y))^2 \tag{6}$$

for nonnegative conductances $c(x, y)$ on the edges of V_0. We will say that $\mathcal{E}_0$ is *nondegenerate* if enough of the conductances are positive so that $\mathcal{E}_0(u) = 0$ if and only if u is constant on V_0. We then extend $\mathcal{E}_0$ to $\mathcal{E}_1$ on V_1 by

$$\mathcal{E}_1(u) = \sum_i r_i^{-1} \mathcal{E}(u \circ F_i). \tag{7}$$

Explicitly,

$$\mathcal{E}_1(u) = \sum_{x \underset{1}{\sim} y} c_1(x, y)(u(x) - u(y))^2 \tag{8}$$

for

$$c_1(F_i x, F_i y) = r_i^{-1} c(x, y) \text{ if } x, y \in V_0. \tag{9}$$

Given the values of u on V_0, we define the *harmonic* extension $\widetilde{u}$ of u to V_1 to be the extension that minimizes $\mathcal{E}_1(u)$. We say that $\mathcal{E}_0$ solves the *renormalization problem* with weights $\{r_i\}$ if

$$\mathcal{E}_1(\widetilde{u}) = \mathcal{E}_0(u) \tag{10}$$

for all u defined on V_0. If this holds, we may define the energy

$$\mathcal{E}_m(u) = \sum_{|w|=m} r_w^{-1} \mathcal{E}_0(u \circ F_w) \tag{11}$$

on Γ_m and show that $\mathcal{E}_m(u)$ is monotone increasing in m, with constant values for harmonic functions (the functions that minimize $\mathcal{E}_m(u)$ for given values on V_0). Then

$$\mathcal{E}(u) = \lim_{m \to \infty} \mathcal{E}_m(u) \tag{12}$$

is well defined as an extended real number, and we may define dom $\mathcal{E}$ as the functions with $\mathcal{E}(u) < \infty$. It can be shown that dom $\mathcal{E}$ is a dense subspace of $C(K)$, so all such functions are determined by their values on $V_* = \bigcup_m V_m$, which is dense in K.

The energy defined above satisfies the self–similar identity Eq. 3 by construction. Conversely, every self–similar energy arises in this manner. Thus the existence of self–similar energy on K is equivalent to the solvability of the renormalization problem, which is just a fixed point problem for a mapping in a finite dimensional space. Note that any solution can be multiplied by a positive constant, so it is better to think of the space of energy forms $\mathcal{E}_0$ on V_0 as a projective space, and to say the solution is *unique* if there is only a single ray of solutions. Likewise it is better to projectivize the set of weights: given projective weights $\{r_i\}$ we try to solve

$$\mathcal{E}_1(\widetilde{u}) = \lambda \mathcal{E}_0(u) \text{ for some } \lambda > 0, \tag{13}$$

and then choose $\{\widetilde{r}_i\} = \{\lambda^{-1} r_i\}$ as the actual weights to define the self–similar energy. Typically there will be at most one possible λ, so every ray of projective weights will contain at most one set of weights for which the renormalization problem is solvable.

There is an extensive literature on the renormalization problem (for example [3, 8–18]). Two types of problems considered are the following:

i) given $\{F_i\}$, does there exist a choice of weights for which the renormalization problem is solvable?
ii) given $\{F_i\}$, characterize those weights for which the problem is solvable, and decide if the solution is unique. Both problems have been completely solved for the case of the Sierpinski gasket (SG), where $F_i = \frac{1}{2}x + \frac{1}{2}q_i, i = 0, 1, 2$, where $\{q_i\}$ are vertices of an equilateral triangle in the plane. In [4] the renormalization problem was solved for projective weights $(1, 1, 1)$. It turns out that $\lambda = 5/3$ is

the unique value in Eq. 13, so the actual weights are $(\frac{3}{5}, \frac{3}{5}, \frac{3}{5})$, and the unique conductances in Eq. 6 are $c(q_i, q_j) = 1$, $i \neq j$. Sabot [18] answered the second question in detail, providing necessary and sufficient conditions on the projective weights for a solution to exist, and showing that it is unique when the conditions hold. (That there are obstructions to existence had been previously known, since the symmetric case $r_1 = r_2$ can be analyzed explicitly by hand.) Sabot's solutions are not entirely explicit, and his proof is rather long and difficult. This paper is a footnote to his work, and the message is: add twists, and everything becomes easy!

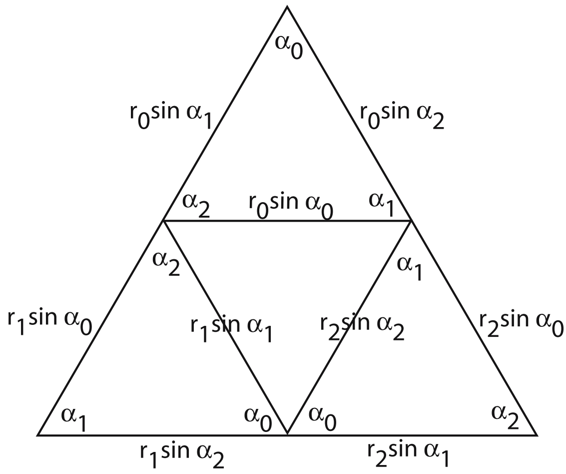
It is well–known that the same fractal may be generated by many different IFSs. For example, one can always take iterations of the original mappings. But if the fractal has symmetries, it may be possible to replace the original IFS by a quite different one. In the case of SG, let R_i denote the reflection that fixes q_i and interchanges the other two boundary points. These are the twists. (See [20] Section 6 and [22] for other contexts in which these twists are important.) Let $\widetilde{F}_i = F_i \circ R_i$. Then $\{\widetilde{F}_i\}$ is an IFS that generates SG. We ask, what are the self–similar energies on SG with respect to this IFS? (Note that these are not the same as the self–similar energies for the original IFS, except in the case of equal weights.) The answer is that for every choice of projective weights there is a unique λ and a unique solution. In Section 2 we give an explicit description of the solution, which involves nothing more complicated than solving an explicit 4th order polynomial equation. (We also show that existence and uniqueness follow from general results in [18].) In Section 3 we show that the set of admissible weights forms a smooth surface in $\mathbb{R}^3$ that is a piece of an explicit algebraic variety, and we write down the 6th order polynomial that defines it. This computation required the use of Macaulay2 [2], but everything else is done by hand. In Section 4 we briefly discuss what happens if we use an IFS with two twists. (Numerical evidence may be found at the website http://www.math.cornell.edu/~cucuringum.) We also show that this does not hold for an IFS with just one twist.

Each self–similar energy gives rise to a self–similar Laplacian (this requires the choice of a self–similar measure, with contraction ratios related naturally to the weights $\{r_i\}$). The spectra of those Laplacians will be discussed in [1].

To give a hint why the IFS with twists should be better than the IFS without twists, we consider the following geometric problem: given a triangle T with vertices q_0, q_1, q_2, can we find similar triangles T_0, T_1, T_2 such that q_j is a vertex of T_j and T_j and T_k intersect at a single point for $j \neq k$? What are the possible similarity ratios r_0, r_1, r_2? If we require the similarities to be orientation preserving, the only possibility is to take $r_0 = r_1 = r_2 = \frac{1}{2}$, and we get an affine image of the usual SG if we form an IFS of the three similarities. However, if we require the similarities to be orientation reversing, then we get an interesting solution for any acute triangle. If $\alpha_0, \alpha_1, \alpha_2$ are the angles at the vertices q_0, q_1, q_2, then we may take $(r_0, r_1, r_2) = (\cos\alpha_0, \cos\alpha_1, \cos\alpha_2)$ as shown in Fig. 1. Indeed, it is easy to check that

$$
\begin{aligned}
r_1 \sin\alpha_2 + r_2 \sin\alpha_1 &= \sin\alpha_0 \\
r_0 \sin\alpha_2 + r_2 \sin\alpha_0 &= \sin\alpha_1 \\
r_0 \sin\alpha_1 + r_1 \sin\alpha_0 &= \sin\alpha_2
\end{aligned}
$$

Fig. 1



using the addition formula for sines and $\alpha_0 + \alpha_1 + \alpha_2 = \pi$. If we take the IFS consisting of these three similarities, the invariant set will be a topological SG with a different geometric structure. It is easy to check that the contraction ratios (r_0, r_1, r_2) that arise in this way satisfy the identity

$$r_0^2 + r_1^2 + r_2^2 + 2r_0r_1r_2 = 1, \tag{14}$$

and conversely any positive solution to Eq. 14 is of the form $(\cos\alpha_0, \cos\alpha_1, \cos\alpha_2)$ for some acute triangle. Of course Eq. 14 defines an algebraic variety, and we are taking the intersection of this variety with the positive octant. It is also easy to see that every ray in the positive octant intersects this variety in a unique point.

2 Existence and Uniqueness

Theorem 2.1 *For any positive projective weights* (r_0, r_1, r_2)*, there exists a unique positive* λ *such that for the weights* $\lambda^{-1}(r_0, r_1, r_2) = (\widetilde{r}_0, \widetilde{r}_1, \widetilde{r}_2)$ *there is a unique (up to a constant multiple) nondegenerate energy* $\mathcal{E}$ *satisfying*

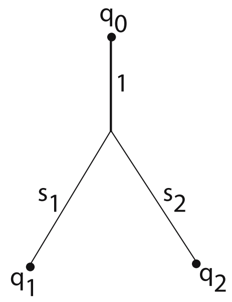
$$\mathcal{E}(u) = \sum_{i=0}^{2} \widetilde{r}_i^{-1}\mathcal{E}(u \circ \widetilde{F}_i). \tag{15}$$

Moreover we have

$$0 < \widetilde{r}_i < 1 \; for\; i = 0, 1, 2. \tag{16}$$

Proof Without loss of generality we may take $r_0 = 1$. By the well-known $\Delta - Y$ transform, we may represent the energy $\mathcal{E}_0$ by the resistance network shown in Fig. 2 (without loss of generality we have taken $s_0 = 1$). Then the energy $\mathcal{E}_1$ is represented by the resistance network shown in Fig. 3. We then transform this network into an equivalent one in the same configuration as Fig. 2, by adding resistances in series and

Fig. 2



using the $\Delta - Y$ transformation. This three stage procedure is shown in Fig. 4, where we use the abbreviation

$$\Sigma = r_1 + r_2 + s_1 + s_2 + r_1 s_2 + r_2 s_1. \tag{17}$$

The renormalization equation says that the resulting network must be a multiple of the original network. This leads to the set of equations.

$$\Sigma + (1 + r_1)(1 + r_2) s_1 s_2 = \lambda \Sigma \tag{18}$$

$$r_1 s_1 \Sigma + (r_1 + r_2)(1 + r_1) s_2 = \lambda s_1 \Sigma \tag{19}$$

$$r_2 s_2 \Sigma + (r_1 + r_2)(1 + r_2) s_1 = \lambda s_2 \Sigma. \tag{20}$$

Note that to have a nondegenerate solution both s_1 and s_2 must be positive. We can use Eq. 18 to determine λ. Since obviously $\lambda > 1$, we conclude $\widetilde{r}_0 = \lambda^{-1} < 1$, so any solution we find will automatically satisfy Eq. 16 (the problem is symmetric so also $\widetilde{r}_1 < 1$ and $\widetilde{r}_2 < 1$).

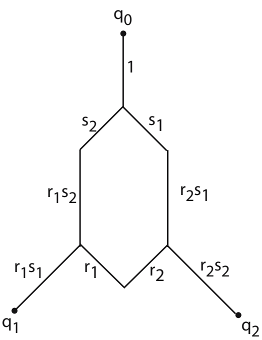
The remaining equations are just

$$r_1 s_1 \Sigma + (r_1 + r_2)(1 + r_1) s_2 = s_1 \Sigma + (1 + r_1)(1 + r_2) s_1^2 s_2 \tag{21}$$

$$r_2 s_2 \Sigma + (r_1 + r_2)(1 + r_2) s_1 = s_2 \Sigma + (1 + r_1)(1 + r_2) s_1 s_2^2. \tag{22}$$

To complete the proof we need to show that for every positive (r_1, r_2) there exists a unique positive solution (s_1, s_2). This is a pair of cubic equations in two variables, but rather luckily we observe that the first equation is linear in s_2, and the second is

Fig. 3



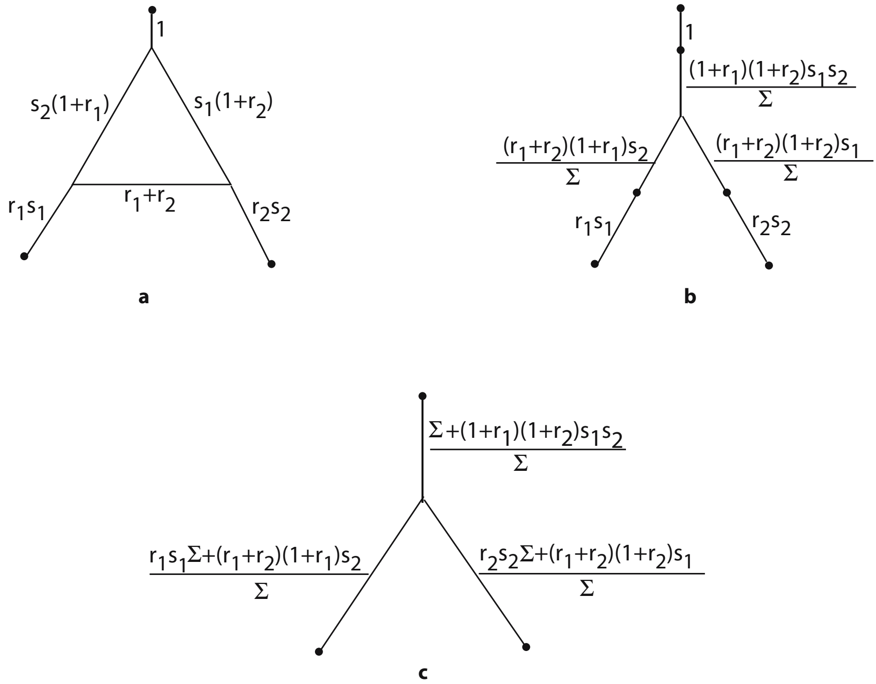


Fig. 4

linear in s_1. (This good luck does not occur without the twists.) So let's solve Eq. 21 for s_2:

$$s_2 = \left(\frac{r_1 - 1}{r_1 + 1}\right) \frac{s_1((1 + r_1)s_1 + (r_1 + r_2))}{(s_1 + 1)((1 + r_2)s_1 - (r_1 + r_2))} \tag{23}$$

provided the denominator does not vanish. Then we substitute Eq. 23 into Eq. 22 and multiply by the denominator to obtain a quartic equation in s_1. For simplicity we drop the subscript and use the abbreviations

$$L = (1 + r_2)s + (r_1 + r_2) \tag{24}$$

$$Q = (s + 1)((1 + r_2)s - (r_1 + r_2)), \tag{25}$$

and the equation becomes

$$\begin{aligned}(1 + r_2)((r_1 - 1)sL + (r_1 + 1)Q)((r_1 - 1)sL - (r_1 + r_2)Q)&\\ + (1 - r_2)(r_1 - 1)L((r_1 - 1)sL + (r_1 + r_2)Q) = 0.&\end{aligned} \tag{26}$$

It is convenient now to consider the three cases $r_1 = 1$, $r_1 < 1$ and $r_1 > 1$. In the case $r_1 = 1$, Eq. 21 simplifies to $2(1 + r_2)s_2 = 2(1 + r_2)s_1^2 s_2$, so $s_1 = \pm 1$. Since we want positive solutions we must have $s_1 = 1$, and then Eq. 22 becomes the quadratic equation

$$4s_2^2 + \left(2 - 2r_2^2\right) s_2 - (1 + r_2)^2 = 0 \tag{27}$$

with two real solutions

$$s_2 = \frac{r_2^2 - 1 \pm \sqrt{\left(1 - r_2^2\right)^2 + 4(1 + r_2)^2}}{4}. \tag{28}$$

Clearly the choice of the plus sign gives the unique positive solution. So that is the complete explicit solution in this case.

In the other two cases, when $r_1 \neq 1$, we will see that the denominator does not vanish in Eq. 23 for the solution of Eq. 26. Indeed, the denominator vanishes exactly when $s_1 = \frac{r_1+r_2}{1+r_2}$, and moreover when $r_1 < 1$ we have $s_2 > 0$ provided $s_1 < \frac{r_1+r_2}{1+r_2}$, while when $r_1 > 1$ we have $s_2 > 0$ provided $s_1 > \frac{r_1+r_2}{1+r_2}$. So our problem has a positive solution provided that Eq. 26 has a solution in $\left(0, \frac{r_1+r_2}{1+r_2}\right)$ when $r_1 < 1$ and in $\left(\frac{r_1+r_2}{1+r_2}, \infty\right)$ when $r_1 > 1$. In fact we will show that for $r_1 \neq 1$, Eq. 26 has exactly one solution in each of these intervals.

The key observation is that we can compute the polynomial $P(s)$ in Eq. 26 exactly at the points $\frac{r_1+r_2}{1+r_2}$, 0, -1 and $\pm\infty$, and see that it changes sign. It then follows that it has zeroes in the two positive intervals and the two negative intervals, and since it is a quartic polynomial the roots in each interval are unique. (It is easy to see that the four roots correspond to all four possible signs $(\pm, \pm)$ for (s_1, s_2).)

The reader who is willing to do a little algebra by hand can easily check the following facts about $P(s)$:

$$P(s) = -2(r_1 + r_2)^4 s^4 + \text{ lower order terms} \tag{29}$$

$$P(-1) = 2r_2(1 - r_1)^4 \tag{30}$$

$$P(0) = -2(r_1 + r_2)^4 \tag{31}$$

$$P\left(\frac{r_1 + r_2}{1 + r_2}\right) = \frac{4(r_1 - 1)^2(1 + r_1)(r_1 + r_2)^3}{(1 + r_2)} \tag{32}$$

(note that $Q = 0$ in Eqs. 30 and 32). This gives the desired result when $r_1 \neq 1$. (Note that when $r_1 = 1$ that P has double roots at $s = \pm 1$.) □

From the form of the solution we see that $r_1 > 1 = r_0$ implies $s_1 > 1 = s_0$, and so the order of the resistances s_0, s_1, s_2 is the same as the order of the weights r_0, r_1, r_2.

We can also express the solution in terms of the initial conductances in

$$\mathcal{E}_0(u) = c_{01}(u(q_0) - u(q_1))^2 + c_{12}(u(q_1) - u(q_2))^2 + c_{20}(u(q_2) - u(q_0))^2. \tag{33}$$

Using the $\Delta - Y$ transform we obtain

$$c_{01} = \frac{s_2}{s_1 + s_2 + s_1 s_2},\ c_{12} = \frac{1}{s_1 + s_2 + s_1 s_2},\ c_{20} = \frac{s_1}{s_1 + s_2 + s_1 s_2}. \tag{34}$$

We could equivalently take

$$c_{01} = s_2,\ c_{12} = 1,\ c_{20} = s_1. \tag{35}$$

We remark that Theorem 2.1 is also a consequence of Theorem 5.1 (ii) of Sabot [18]. To see this we observe (in the notation of [18]) that the only nontrivial preserved

G–relations are the one that identifies q_1 and q_2, and permutations of this one. Thus assumption (H) is valid. We need to verify $\overline{\rho}_J < \underline{\rho}_{F/J}$ for this relation (the computation for the others is identical). In this case, both T_J and $T_{F/J}$ are multiplies of the identity, so both ρ values are just the multiples. For F/J, the multiple is r_0, while for J it is $\left(\frac{1}{r_0}+\frac{1}{r_1}+\frac{1}{r_2}\right)^{-1}$, so the inequality is obvious. The same argument works for the case of two twists (in that case there is only one preserved G-relation, and the computations are identical). Of course this is just an abstract existence and uniqueness statement, and doesn't yield the explicit formulas we have found.

3 The Surface of Admissible Weights

Let us call the actual weights $(\widetilde{r}_0, \widetilde{r}_1, \widetilde{r}_2)$ that arise in Theorem 2.1 *admissible weights*. For every ray in the positive octant, there is a unique admissible weight lying on it. Thus the admissible weights from a surface in $\mathbb{R}^3$. In this section we show that it is smooth (C^∞) and is part of an algebraic variety. Let

$$\begin{cases} G_1(r_1,r_2,s_1,s_2) = (r_1-1)(r_1+r_2)s_1 + (r_1-1)(1+r_2)s_1^2 \\ \qquad + (r_1-1)(1+r_1)s_1s_2 + (r_1+r_2)(1+r_1)s_2 \\ \qquad - (1+r_1)(1+r_2)s_1^2s_2, \\ G_2(r_1,r_2,s_1,s_2) = (r_2-1)(r_1+r_2)s_2 + (r_2-1)(1+r_1)s_2^2 \\ \qquad + (r_2-1)(1+r_2)s_1s_2 + (r_1+r_2)(1+r_2)s_1 \\ \qquad - (1+r_1)(1+r_2)s_1s_2^2. \end{cases} \tag{36}$$

Then the equations

$$\begin{cases} G_1(r_1,r_2,s_1,s_2) = 0 \\ G_2(r_1,r_2,s_1,s_2) = 0 \end{cases} \tag{37}$$

describe how the variables (r_1, r_2) determine the variables (s_1, s_2), and the surface of admissible weights is given by

$$(x, y, z) = (\lambda^{-1}, \lambda^{-1}r_1, \lambda^{-1}r_2) \tag{38}$$

for

$$\lambda^{-1} = \frac{\Sigma}{\Sigma + (1+r_1)(1+r_2)s_1s_2} \tag{39}$$

(Σ is given by Eq. 17).

Lemma 3.1 (s_1, s_2) *are smooth functions of* (r_1, r_2).

Proof If $r_1 \neq 1$ and $r_2 \neq 1$, then Eq. 23 shows that s_2 is a smooth function of (s_1, r_1, r_2), and s_1 is a smooth function of (r_1, r_2) since it is a solution of the quartic equation Eq. 26 that has four distinct roots. To show that the solution remains smooth when $r_1 = 1$ or $r_2 = 1$ we will use the implicit function theorem. For this it suffices to show that the determinant of the matrix

$$\begin{pmatrix} \frac{\partial G_1}{\partial s_1} & \frac{\partial G_1}{\partial s_2} \\ \frac{\partial G_2}{\partial s_1} & \frac{\partial G_2}{\partial s_2} \end{pmatrix} \tag{40}$$

is never zero. We compute

$$\begin{aligned} \frac{\partial G_1}{\partial s_1} &= (r_1 - 1)(r_1 + r_2) + 2(r_1 - 1)(1 + r_2)s_1 + (r_1 - 1)(1 + r_1)s_2 \\ &\quad -2(1 + r_1)(1 + r_2)s_1 s_2 \\ \frac{\partial G_1}{\partial s_2} &= (r_1 - 1)(1 + r_1)s_1 + (r_1 + r_2)(1 + r_1) - (1 + r_1)(1 + r_2)s_1^2 \\ \frac{\partial G_2}{\partial s_1} &= (r_2 - 1)(1 + r_2)s_2 + (r_1 + r_2)(1 + r_2) - (1 + r_1)(1 + r_2)s_2^2 \\ \frac{\partial G_2}{\partial s_2} &= (r_2 - 1)(r_1 + r_2) + 2(r_2 - 1)(1 + r_1)s_2 + (r_2 - 1)(1 + r_2)s_1 \\ &\quad -2(1 + r_1)(1 + r_2)s_1 s_2. \end{aligned}$$

Now we note that when $r_1 = 1$ (hence $s_1 = 1$) we have $\frac{\partial G_1}{\partial s_2} = 0$, so the determinant of Eq. 40 is just

$$\frac{\partial G_1}{\partial s_1} \frac{\partial G_2}{\partial s_2} = \left(-2(1 + r_2)^2 s_2\right)(2(r_2 - 1)(1 + r_2) - 8s_2). \tag{41}$$

This can only vanish if $s_2 = \frac{1}{4}\left(r_2^2 - 1\right)$, but this contradicts Eq. 28. By symmetry, a similar argument works if $r_2 = 1$. □

Theorem 3.2 *The surface of admissible weights is smooth.*

Proof By Lemma 3.1, (x, y, z) are smooth functions of the parameters (r_1, r_2). By the implicit function theorem it suffices to show that the matrix

$$\begin{pmatrix} \frac{\partial x}{\partial r_1} & \frac{\partial y}{\partial r_1} & \frac{\partial z}{\partial r_1} \\ \frac{\partial x}{\partial r_2} & \frac{\partial y}{\partial r_2} & \frac{\partial z}{\partial r_2} \end{pmatrix} \tag{42}$$

has rank 2 at every point. Now the three determinants of the 2×2 minors of Eq. 42 are

$$\left(\lambda^{-1} + r_1\frac{\partial\lambda^{-1}}{\partial r_1}\right)\left(\lambda^{-1} + r_2\frac{\partial\lambda^{-1}}{\partial r_2}\right) - r_1 r_2\frac{\partial\lambda^{-1}}{\partial r_1}\frac{\partial\lambda^{-1}}{\partial r_2} = \lambda^{-1}\left(\lambda^{-1} + r_1\frac{\partial\lambda^{-1}}{\partial r_1} + r_2\frac{\partial\lambda^{-1}}{\partial r_2}\right),$$

$$r_1\frac{\partial\lambda^{-1}}{\partial r_1}\frac{\partial\lambda^{-1}}{\partial r_2} - \frac{\partial\lambda^{-1}}{\partial r_2}\left(\lambda^{-1} + r_1\frac{\partial\lambda^{-1}}{\partial r_1}\right) = -\lambda^{-1}\frac{\partial\lambda^{-1}}{\partial r_2},$$

$$r_2\frac{\partial\lambda^{-1}}{\partial r_1}\frac{\partial\lambda^{-1}}{\partial r_2} - \frac{\partial\lambda^{-1}}{\partial r_1}\left(\lambda^{-1} + r_2\frac{\partial\lambda^{-1}}{\partial r_2}\right) = -\lambda^{-1}\frac{\partial\lambda^{-1}}{\partial r_1}.$$

But if the last two vanish then we must have $\frac{\partial\lambda^{-1}}{\partial r_1} = \frac{\partial\lambda^{-1}}{\partial r_2} = 0$, and then the first one cannot vanish. □

Because G_1 and G_2 are polynomials and (x, y, z) are rational functions, it follows easily that Eq. 38 defines an algebraic variety. Our surface is a subset of this variety specified by the positivity of r_1, r_2, s_1, s_2. By symmetry, the polynomial equation of this variety must be symmetric, hence expressible in terms of the elementary symmetric polynomials

$$e_1 = x + y + z,\ \ e_2 = xy + yz + zx,\ \ e_3 = xyz. \tag{43}$$

Theorem 3.3 *The surface of admissible weights is a subset of the symemtric order 6 algebraic variety given by*

$$\begin{aligned} &-e_1^3 e_2 + e_1^3 e_3 - e_1^2 e_2 + 4e_1 e_2^2 - 3e_1^2 e_3 - 4e_1 e_2 e_3 + e_1 e_2 + 7e_1 e_3 \\ &-4e_2 e_3 + 4e_3^2 + e_2 - 5e_3 = 0. \end{aligned} \tag{44}$$

Proof Equation 44 was found using Macaulay2. A worksheet may be found on the website. □

A graph of the admissible weights surface is shown in Fig. 5. The data was generated by solving Eqs. 21 and 22 for various inputs of r values, not by solving Eq. 44.

A related question is whether or not there are constraints on the s values that arise. It is not difficult to see that there must be constraints just by considering the diagonal case $s_1 = 1$, for then $s_2 > \frac{\sqrt{5}-1}{4}$ is required. In fact we can give a complete description of the *admissible s–region* of all values (s_1, s_2) in the plane which correspond to solutions. One boundary curve of the region is obtained by letting $r_1 \to 0^+$. Setting $r_1 = 0$ in Eqs. 21 and 22 we obtain

$$r_2\left(s_2 - s_1 - s_1^2 - s_1^2 s_2\right) = s_1^2 + s_1 s_2 + s_1^2 s_2 \tag{45}$$

$$r_2^2(s_2 + s_1 s_2 + s_1) + r_2\left(s_2^2 - s_2 + s_1 - s_1 s_2^2\right) = s_2^2 + s_1 s_2 + s_1 s_2^2. \tag{46}$$

Note that from Eq. 45 we deduce that

$$s_1 < \frac{s_2}{1 + s_2} \text{ (this implies } s_1 < 1) \tag{47}$$

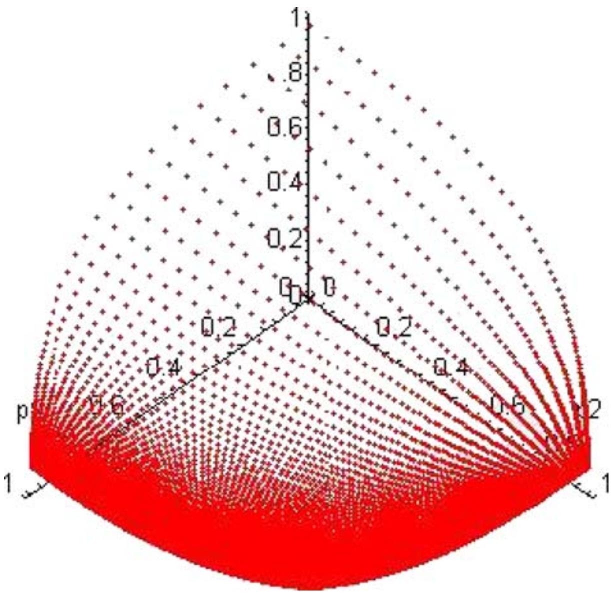


Fig. 5 A graph of the surface of admissible weights $(\widetilde{r}_0, \widetilde{r}_1, \widetilde{r}_2)$

is necessary for $r_2 \geq 0$. When we eliminate r_2 from Eqs. 45 and 46 we obtain

$$-s_1^3 s_2^3 + s_1^3 s_2^2 + s_1^2 s_2^3 + 2s_1^2 s_2^2 + s_1^3 s_2 + s_1 s_2^3 - s_1^3 + s_1^2 s_2 + s_1 s_2^2 - s_2^3 = 0. \tag{48}$$

It is remarkable that Eq. 48 is symmetric in s_1 and s_2. Thus Eq. 48 also describes the boundary $r_2 = 0$, but instead of Eq. 47 we have $s_2 < 1$. To find the third boundary curve corresponding to $r_0 \to 0^+$ we need to substitute $(s_1, s_2) \to (1/s_1, s_2/s_1)$ in Eq. 48. Remarkably, this just yields Eq. 48 again. So the same algebraic curve describes the entire boundary of the admissible s–region. If we let $Q(s_1, s_2)$ denote the left side of Eq. 48, then the admissible s–region is described by

$$Q(s_1, s_2) > 0, \tag{49}$$

since $(s_1, s_2) = (1, 1)$ belongs to the region and satisfies Eq. 49. This region is shown in Figs. 6 and 7.

Another way of looking at it is that we should consider the cone in the positive octant in $\mathbb{R}^3$ of all values (s_0, s_1, s_2) that arise, without the normalization $s_0 = 1$. Then the admissible s–region is the intersection of the cone with the plane $s_0 = 1$. To get the equation for the cone we just have to homogenize Eq. 48. We obtain

$$\begin{aligned} &2s_0^2 s_1^2 s_2^2 - \left(s_1^3 s_2^3 - s_0^3 s_1^3 + s_0^3 s_2^3\right) \\ &\quad + \left(s_0 s_1^2 s_2^3 + s_0 s_1^3 s_2^2 + s_0^2 s_1 s_2^3 + s_0^2 s_1^3 s_2 + s_0^3 s_1 s_2^2 + s_0^3 s_1^2 s_2\right) > 0. \end{aligned} \tag{50}$$

In terms of the elementary symmetric polynomials e_1, e_2, e_3 (in the variables s_0, s_1, s_2) this may be written

$$4e_1 e_2 e_3 - e_2^3 - 4e_3^2 > 0. \tag{51}$$

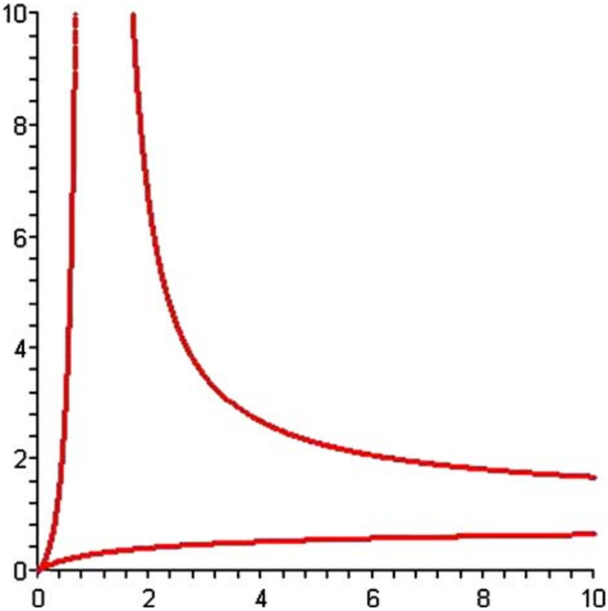


Fig. 6 The admissible s-region, bounded by three branches of the curve Eq. 48

4 The Case of Two Twists

In this section we briefly discuss self–similar energies with respect to the IFS $\{F_0, \widetilde{F}_1, \widetilde{F}_2\}$. Note that we use exactly two twists, rather than three. As mentioned in Section 2, the results of [18] imply the following: for any positive projective weights (r_0, r_1, r_2) there exists a unique positive λ' such that for the weights $(\lambda')^{-1}(r_0, r_1, r_2) =$

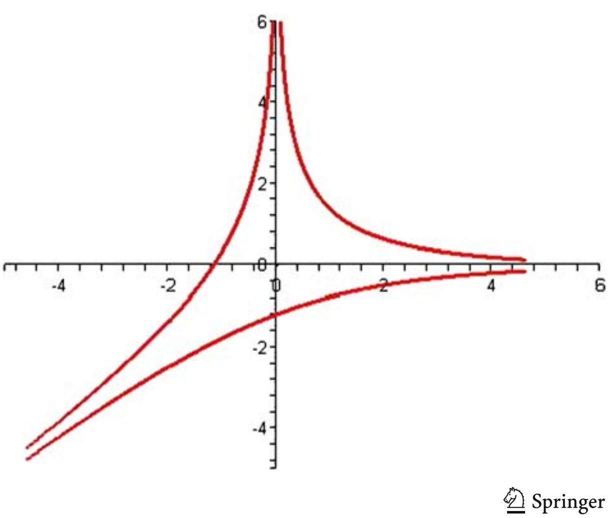


Fig. 7 The same region as in Fig. 6, but on a logarithmic scale $(\log s_1, \log s_2)$

(r_0', r_1', r_2') there is a unique (up to a constant multiple) nondegenerate energy $\mathcal{E}'$ satisfying

$$\mathcal{E}'(u) = \left(r_0'\right)^{-1} \mathcal{E}'(u \circ F_0) + \left(r_1'\right)^{-1} \mathcal{E}'\left(u \circ \widetilde{F}_1\right) + \left(r_2'\right)^{-1} \mathcal{E}'\left(u \circ \widetilde{F}_2\right), \tag{52}$$

and

$$0 < r_i' < 1 \text{ for } i = 0, 1, 2. \tag{53}$$

We would also like to have an explicit formula in this case.

If we follow the argument in the proof of Theorem 2.1 we obtain a slightly different set of equations. In Fig. 3 we have to interchange the conductances s_1 and s_2 at the top of the hexagon. In place of Eqs. 18, 19, and 20 we obtain

$$\Sigma + (s_1 + r_1 s_2)(s_2 + s_1 r_2) = \lambda' \Sigma \tag{54}$$

$$r_1 s_1 \Sigma + (r_1 + r_2)(s_1 + r_1 s_2) = \lambda' s_1 \Sigma \tag{55}$$

$$r_2 s_2 \Sigma + (r_1 + r_2)(s_2 + r_2 s_1) = \lambda' s_2 \Sigma, \tag{56}$$

where Σ is still given by Eq. 17. We let Eq. 54 define λ', and then substitute into Eqs. 55 and 56 to obtain

$$r_1 s_1 \Sigma + (r_1 + r_2)(s_1 + r_1 s_2) = s_1 \Sigma + s_1(s_1 + r_1 s_2)(s_2 + r_2 s_1) \tag{57}$$

$$r_2 s_2 \Sigma + (r_1 + r_2)(s_2 + r_2 s_1) = s_2 \Sigma + s_2(s_1 + r_1 s_2)(s_2 + r_2 s_1), \tag{58}$$

in place of Eqs. 21 and 22. However, we have not been able to solve this system explicitly, as neither Eq. 57 nor Eq. 58 is linear in either s–variable.

Instead, we have solved the system Eqs. 57 and 58 numerically for many choices of the r–variables. We further computed λ' and the admissible weights $\left(r_0', r_1', r_2'\right)$ for this problem. A graph of this surface is shown in Fig. 8. This is the analog of Fig. 5. See the website www.math.cornell.edu/~cucuringum for details.

We note that in the case that $r_1 = r_2$, the problem is symmetric and the two-twist and three-twist problems are equivalent, so the s-values are the same and $\lambda' = \lambda$. In general, we do not expect that the solutions of the two problems will be identical. Similar reasoning shows that, in the symmetric case $r_1 = r_2$, the one-twist IFS $\left\{\widetilde{F}_0, F_1, F_2\right\}$ and the zero-twist IFS $\{F_0, F_1, F_2\}$ yield identical problems. Since it is known that the zero-twist IFS does not allow solutions for $r_1 = r_2 \leq \frac{2}{3}$, we conclude that the analog of Theorem 2.1 does not hold for the one-twist IFS.

We also note that the geometric problem discussed at the end of the introduction also has a solution for any acute triangle if we require exactly two of the similarities to be orientation reversing. In this case the only change in Fig. 1 is that the top triangle is reflected, so the equations become

$$r_1 \sin \alpha_2 + r_2 \sin \alpha_1 = \sin \alpha_0$$
$$r_0 \sin \alpha_1 + r_2 \sin \alpha_0 = \sin \alpha_1$$
$$r_0 \sin \alpha_2 + r_1 \sin \alpha_0 = \sin \alpha_2,$$

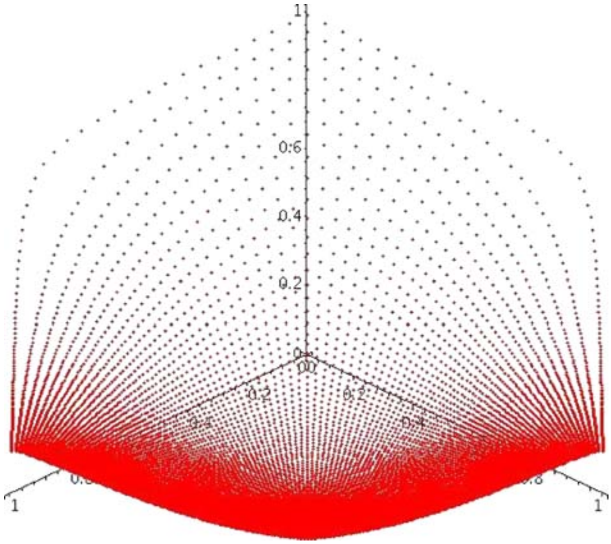


Fig. 8 A graph of the surface of admissible weights (r'_0, r'_1, r'_2)

and the unique solution is

$$r_0 = \frac{\sin^2\alpha_1 + \sin^2\alpha_2 - \sin^2\alpha_0}{\sin^2\alpha_1 + \sin^2\alpha_2}$$

$$r_1 = \frac{\sin\alpha_0 \sin\alpha_2}{\sin^2\alpha_1 + \sin^2\alpha_2}$$

$$r_2 = \frac{\sin\alpha_0 \sin\alpha_2}{\sin^2\alpha_1 + \sin^2\alpha_2}.$$

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