Lecture 16: Group synchronization and applications

Foundations of Data Science: Algorithms and Mathematical Foundations

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Recovering signal from pairwise noisy comparisons

- let $r = (r_1, \ldots, r_n)^T \in \mathbb{R}^n$ be an unknown signal (for eg, unknown latent strength of a player)
- $G = ([n], E)$ is an undirected measurement graph
- we are given a subset of noisy pairwise measurements

$$M_{ij} = r_i - r_j, \quad \text{for each } \{i, j\} \in E \tag{1}$$

  (for eg, results of a match outcome reflecting the skill difference)
- goal: estimate the original vector r
- clearly, only possible only up to a global shift
- when measurements are exact without any measurement noise, one can recover the strength vector $r \iff$ graph G is connected
  - simply consider a spanning tree of G
  - fix the value of the root node
  - traverse the tree & propagate information by summing the offsets
- for simplicity, we assume the graph is connected, (otherwise it is not possible to estimate the offset values between nodes belonging to different connected components of the graph)
- how would you solve this problem?

Synchronization over the real line $\mathbb{R}$

Instantiations of the above problem are ubiquitous in

- engineering
- machine learning
- computer vision
- have received a great deal of attention in the recent literature

Synchronization over the real line $\mathbb{R}$:

- *Time synchronization of wireless networks.* A popular application arises in engineering, and is known as time synchronization of distributed networks where clocks measure noisy time offsets $r_i - r_j$, and the goal is to recover $r_1, \ldots, r_n \in \mathbb{R}$.
- *Ranking.* A fundamental problem in information retrieval is that of recovering the ordering induced by the latent strengths or scores $r_1, \ldots, r_n \in \mathbb{R}$ of a set of n players, that is best reflected by the given set of pairwise comparisons $r_i - r_j$.

- We can frame this problem as that of recovering elements of a group

$$\mathcal{G} = \mathbb{R} : \qquad r_1, \ldots, r_n, \text{ with } r_i \in \mathbb{R} \tag{2}$$

given a small subset of pairwise differences.

- But what about other groups $\mathcal{G}$?

Synchronization over $\mathbb{Z}_2$

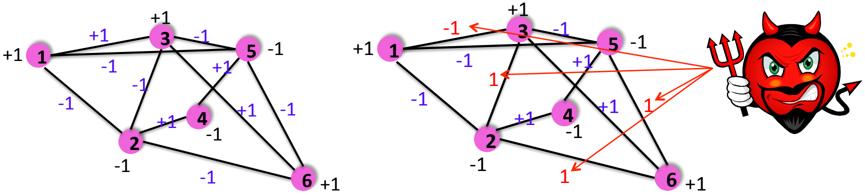


Figure: Synchronization over $\mathbb{Z}_2$ (left: clean, right: noisy)

- unknown group elements $z_1, z_2, \ldots, z_N \in \mathbb{Z}_2$ correspond to the vertices of a measurement graph G
- each edge (i, j) in $E(G)$ holds a noisy version of the ratio of the elements from its endpoints (in $\mathbb{Z}_2$, recall that $z_i = z_i^{-1}$)
- a potential noise model for the measurement graph is

$$Z_{ij} = \begin{cases} z_i z_j^{-1} = z_i z_j & (i,j) \in E \text{ and the measurement is correct,} \\ -z_i z_j^{-1} = -z_i z_j & (i,j) \in E \text{ and the measurement is incorrect,} \\ 0 & (i,j) \notin E \end{cases}$$

- original solution: $z_1, \ldots, z_n \in \pm 1^n \quad (\mathbb{Z}_2 = \{-1, +1\})$
- task: estimate approximated solution $x_1, \ldots, x_N \in \pm 1^N$ such that we satisfy as many pairwise group relations in $\mathbb{Z}_2$ as possible.

Synchronization over $\mathbb{Z}_2$

Consider maximizing the following quadratic form (happy edges)

$$\max_{x_1,\ldots,x_N \in \mathbb{Z}_2^N} \sum_{i,j=1}^{N} x_i Z_{ij} x_j = \max_{x_1,\ldots,x_N \in \mathbb{Z}_2^N} x^T Z x,$$

whose maximum is attained when $x = z$ (noise-free data).

NP-hard problem, but relax to

$$\max_{\sum_{i=1}^{N} |x_i|^2 = N} \sum_{i,j=1}^{N} x_i Z_{ij} x_j = \max_{\|x\|^2 = N} x^T Z x$$

whose maximum is achieved when $x = v_1$, the normalized top eigenvector of Z

$$Z v_1 = \lambda_1 v_1$$

Alternative formulation - Synchronization over $\mathbb{Z}_2$

Start by formulating the synchronization problem as a least squares problem, by minimizing the following quadratic form (unhappy edges)

$$
\begin{aligned}
\min_{x \in \mathbb{Z}_2^N} \sum_{(i,j)\in E} (x_i - Z_{ij}x_j)^2 &= \min_{x \in \mathbb{Z}_2^N} \sum_{(i,j)\in E} x_i^2 + Z_{ij}^2 x_j^2 - 2Z_{ij}x_i x_j \\
&= \min_{x \in \mathbb{Z}_2^N} \sum_{(i,j)\in E} x_i^2 + x_j^2 - 2Z_{ij}x_i x_j \\
&= \min_{x \in \mathbb{Z}_2^N} \sum_{i=1}^{n} d_i x_i^2 - \sum_{(i,j)\in E} 2Z_{ij}x_i x_j \\
&= \min_{x \in \mathbb{Z}_2^N} x^T \overline{D} x - x^T Z x \\
&= \min_{x \in \mathbb{Z}_2^N} x^T (\overline{D} - Z) x
\end{aligned}
$$

- Signed Graph Laplacian

$$
\overline{L} = \overline{D} - Z \tag{3}
$$

where $\overline{D}_{ii} = \sum_{j=1}^{n} |Z_{ij}|$. For the rest of the slides, we use D to denote $\overline{D}$

The Eigenvector Method - noiseless case

Exercise

Claim: One can recover the correct sign (ie, group element in $\mathbb{Z}_2$) at each node from the top eigenvector of $\mathcal{Z} = D^{-1}Z$

- $\mathcal{Z} = D^{-1}Z$
- Diagonal matrix Υ, $\Upsilon_{ii} = z_i$ (ground truth value)
- $A = (a_{ij})$ adjacency matrix of the measurement graph
- Write $Z = (z_{ij})$ as $Z = \Upsilon A \Upsilon^{-1}$, for noiseless data $z_{ij} = z_i z_j$
- $\mathcal{Z} = \Upsilon(D^{-1}A)\Upsilon^{-1}$.
- $\mathcal{Z}$ and $D^{-1}A$ all have the same eigenvalues (similar matrices)
- Normalized discrete graph Laplacian $\mathcal{L} = I - D^{-1}A$
- $I - \mathcal{Z}$ and $\mathcal{L}$ have the same eigenvalues
- $1 - \lambda_i^{\mathcal{Z}} = \lambda_i^{\mathcal{L}} \geq 0$, and $v_i^{\mathcal{Z}} = \Upsilon v_i^{\mathcal{L}}$
- G connected $\Rightarrow \lambda_1^{\mathcal{L}} = 0$ is simple, $v_1^{\mathcal{L}} = \mathbf{1} = (1, 1, \ldots, 1)^T$
- $v_1^{\mathcal{Z}} = \Upsilon \mathbf{1}$ and thus $v_1^{\mathcal{Z}}(i) = z_i$

Synchronization over SO(2)

Estimate n unknown angles (group elements in SO(2))

$$\theta_1, \ldots, \theta_n \in [0, 2\pi),$$

given m noisy measurements δij of their pairwise offsets

$$\delta_{ij} = \theta_i - \theta_j \mod 2\pi. \tag{4}$$

Challenges:

- amount of noise in the measurements, ie, in reality we measure
$$\delta_{ij} = (\theta_i - \theta_j + \text{Noise}) \mod 2\pi. \tag{5}$$
- only a very small subset of all possible pairwise offsets are measured ($m << \binom{n}{2}$)

Questions

- In the noiseless setting, how can we get a solution?
- In general, is the solution unique?

• Chapter 10, *"Synchronization Problems and Alignment"*, in *Ten Lectures and Forty-Two Open Problems in the Mathematics of Data Science*, by Afonso Bandeira

Angular embedding

- A. Singer (2011), spectral and SDP relaxation for the angular synchronization problem
- S. Yu (2012), spectral relaxation; robust to noise when applied to an image reconstruction problem

- embedding in the angular space is significantly more robust to outliers compared to embedding in the usual linear space

Noise models

- many possible models for the measurement errors
- including ones that allow for many outliers
- an outlier is an offset measurement that has a uniform distribution on $[0, 2\pi)$, regardless of the true value for the offset
- there also exist (of course) good measurements whose errors are relatively small (or even zero error; assume this for simplicity)
- the user has no a-priori knowledge on which measurements are good and which are bad (outliers)

- the edge set E can be split into
- a set of good edges E_{good} of size m_{good}
- a set of bad edges E_{bad} of size m_{bad}
- with $m = |E| = m_{good} + m_{bad}$, s.t.

$$\begin{array}{ll} \delta_{ij} = \theta_i - \theta_j & \text{for } \{i,j\} \in E_{good} \\ \delta_{ij} \sim Uniform([0, 2\pi)) & \text{for } \{i,j\} \in E_{bad} \end{array} . \tag{6}$$

Least-squares approach for SO(2)

- over-determined system of linear equations (modulo 2π)

$$\theta_i - \theta_j = \delta_{ij} \mod 2\pi, \quad \text{for } \{i,j\} \in E \tag{7}$$

- can solve via the method of least-squares
- introduce the complex-valued variables $z_i = e^{\imath\theta_i}$
- the system (7) is equivalent to

$$z_i - e^{\imath\delta_{ij}} z_j = 0, \quad \text{for } \{i,j\} \in E, \tag{8}$$

  overdetermined system of homogeneous linear equations over $\mathbb{C}$
- set $z_1 = 1$ (ie. $\theta_1 = 0$) to prevent the solution from collapsing to the trivial solution $z_1 = z_2 = \ldots = z_n = 0$
- find solution $z_2, \ldots, z_n$ of (8) with minimal ℓ_2-norm residual
- least-squares method will be affected by the outliers (as outlier equations will dominate the sum of squares)
- will compare to the least-squares baseline in the simulations
- seek for an alternative solution, more robust to outliers

Towards a spectral relaxation for SO(2)

- build the $n \times n$ sparse Hermitian matrix H

$$H_{ij} = \begin{cases} e^{\imath\delta_{ij}} & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E. \end{cases} \quad (9)$$

- consider the following maximization problem

$$\underset{\theta_1,\ldots,\theta_n \in [0,2\pi)}{\text{maximize}} \sum_{i,j=1}^{n} e^{-\imath\theta_i} H_{ij} e^{\imath\theta_j} \quad (10)$$

- gets incremented by $+1$ whenever an assignment of angles θ_i and θ_j perfectly satisfies the given edge constraint $\delta_{ij} = \theta_i - \theta_j \mod 2\pi$ (i.e., for a *good* edge), since

$$e^{-\imath\theta_i} e^{\imath\delta_{ij}} e^{\imath\theta_j} = e^{-\imath\theta_i} e^{\imath(\theta_i - \theta_j)} e^{\imath\theta_j} = e^0 = +1$$

- the contribution of an incorrect assignment (i.e., of a *bad* edge) will be uniformly distributed on the unit circle

Spectral relaxation

Spectral relaxation given by

$$\underset{z_1,\ldots,z_n\in\mathbb{C};\ \sum_{i=1}^n |z_i|^2=n}{\text{maximize}} \quad \sum_{i,j=1}^{n} \overline{z_i} H_{ij} z_j \tag{11}$$

- where we replaced the individual constraints $z_i = e^{\iota\theta_i}$ having unit magnitude by the much weaker single constraint

$$\sum_{i=1}^{n} |z_i|^2 = n$$

- maximization of a quadratic form

$$\underset{||z||^2=n}{\text{maximize}}\ z^* H z \tag{12}$$

solved for $z = v_1$, the top eigenvector of H

Angular Synchronization

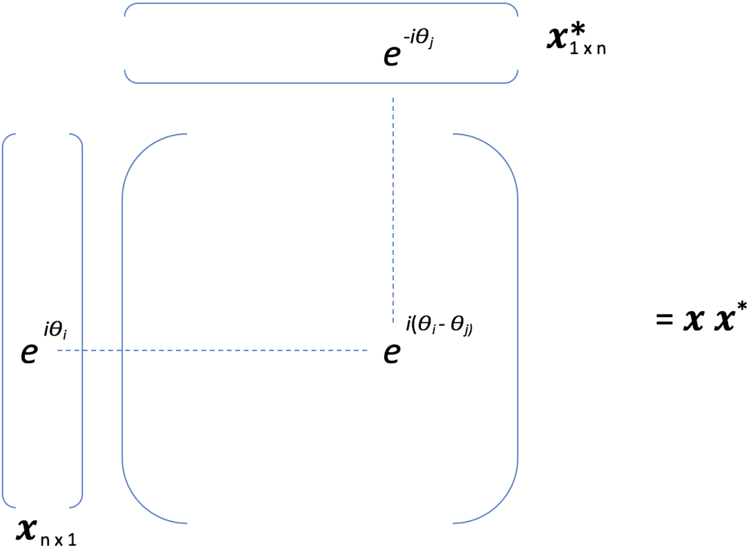


Figure: Rank-1 matrix in the angular domain

The eigenvector magic

- cycles in the graph of good edges E_{good} lead to consistency relations between the offset measurements
- for eg., in a triangle of good edges $\{i,j\},\{j,k\},\{k,i\} \in E_{good}$
  - the corresponding offset angles δ_{ij}, δ_{jk} and δ_{ki} must satisfy

$$\delta_{ij} + \delta_{jk} + \delta_{ki} = 0 \mod 2\pi, \quad \text{since} \tag{13}$$

$$\delta_{ij} + \delta_{jk} + \delta_{ki} = \theta_i - \theta_j + \theta_j - \theta_k + \theta_k - \theta_i = 0 \mod 2\pi \tag{14}$$

- recall the power iteration method
  - multiplying the matrix H by itself integrates the information in the consistency relation of triplets
  - higher order iterations exploit consistency relations of longer cycles

$$H_{ij}^2 = \sum_{k=1}^{n} H_{ik} H_{kj} = \sum_{k:\{i,k\},\{j,k\}\in E} e^{\imath\delta_{ik}} e^{\imath\delta_{kj}} = \sum_{k:\{i,k\},\{j,k\}\in E} e^{-\imath(\delta_{jk}+\delta_{ki})} \tag{15}$$

$$= \#\left\{k : \{i,k\} \text{ and } \{j,k\} \in E_{good}\right\} e^{\imath(\theta_i-\theta_j)} \tag{16}$$

$$+ \sum_{k:\{i,k\} \text{ or } \{j,k\}\in E_{bad}} e^{-\imath(\delta_{jk}+\delta_{ki})},$$

- using $\delta_{ji} = -\delta_{ij}$ in (15), and (13) in (16).
- $\Rightarrow$ the top eigenvector integrates consistency relations of all cycles

Noise model SO(2) (Singer 2011)

- measurement graph G is Erdős-Rényi $G(n, \alpha)$
- each available measurement is either correct with probably p or a random measurement with probability $1 - p$

$$\Theta_{ij} = \begin{cases} \theta_i - \theta_j & \text{for a correct edge} \quad \text{w.p. } p\alpha \\ \sim Uniform(S^1) & \text{for an incorrect edge} \quad \text{w.p. } (1-p)\alpha \\ 0 & \text{for a missing edge,} \quad \text{w.p. } 1-\alpha. \end{cases} \tag{17}$$

- for $G = K_n$ (thus $\alpha = 1$), the spectral relaxation for the angular synchronization problem
  - undergoes a phase transition phenomenon
  - top eigenvector of H exhibits above random correlations with the ground truth solution as soon as

$$p > \frac{1}{\sqrt{n}} \tag{18}$$

  - can be extended to the general Erdős-Rényi case

Amit Singer, *Angular synchronization by eigenvectors and semidefinite programming*, Applied and computational harmonic analysis (2011)

Spectral relaxation

- normalize H by the diagonal matrix D with $D_{ii} = \sum_{j=1}^{n} |H_{ij}|$

$$\mathcal{H} = D^{-1}H, \tag{19}$$

- similar to the Hermitian matrix $D^{-1/2}HD^{-1/2}$ from

$$\mathcal{H} = D^{-1/2}(D^{-1/2}HD^{-1/2})D^{1/2}$$

- $\mathcal{H}$ has n real eigenvalues $\lambda_1^{\mathcal{H}} > \lambda_2^{\mathcal{H}} \geq \cdots \geq \lambda_n^{\mathcal{H}}$ and n orthogonal (complex valued) eigenvectors $v_1^{\mathcal{H}}, \ldots, v_n^{\mathcal{H}}$
- estimated rotation angles $\hat{\theta}_1, ..., \hat{\theta}_n$ using the top eigenvector $v_1^{\mathcal{H}}$ via

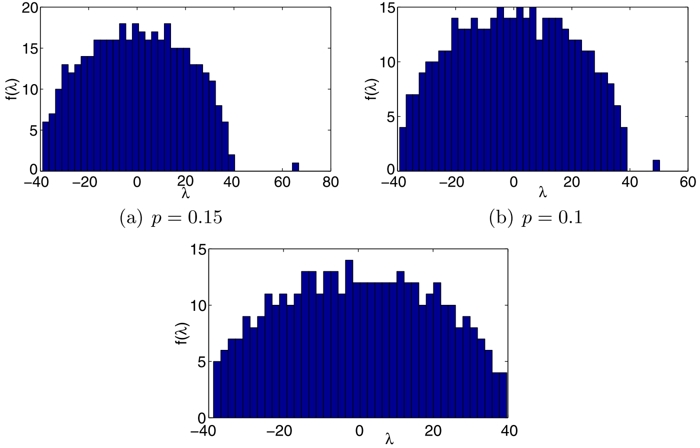
$$e^{\iota\hat{\theta}_i} = \frac{v_1^{\mathcal{H}}(i)}{|v_1^{\mathcal{H}}(i)|}, \quad i = 1, 2, \ldots, n. \tag{20}$$

- up to an additive phase, since $e^{i\alpha}v_1^{\mathcal{H}}$ is also an eigenvector of $\mathcal{H}$ for any $\alpha \in \mathbb{R}$

Spectrum of H

Consider the $n \times n$ sparse Hermitian matrix H

$$H_{ij} = \begin{cases} e^{\imath \delta_{ij}} & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E. \end{cases} \tag{21}$$



(a) $p = 0.15$ (b) $p = 0.1$ (c) $p = 0.05$

Figure: Histograms of the eigenvalues of the matrix H in the complete graph model for $n = 400$ and different values of the noise level p.

Analysis of the complete graph angular synchronization (i)

- H_{ij}'s are random variables given by the following mixture model
  - w. prob. p the edge $\{i, j\}$ is good and $H_{ij} = e^{\imath(\theta_i - \theta_j)}$
  - w. prob. $1 - p$ the edge is bad and $H_{ij} \sim Uniform\left(S^1\right)$
- for convenience, define the diagonal elements as $H_{ii} = p$
- H is Hermitian and the expected value of its elements is

$$\mathbb{E}H_{ij} = p\, e^{\imath(\theta_i - \theta_j)}. \tag{22}$$

- ie., the expected value of H is the rank-one matrix

$$\mathbb{E}H = npzz^*, \tag{23}$$

- where z is the normalized vector ($||z|| = 1$) given by

$$z_i = \frac{1}{\sqrt{n}}\, e^{\imath\theta_i}, \quad i = 1, \ldots, n. \tag{24}$$

Analysis of the complete graph angular synchronization (ii)

- matrix H can be decomposed as

$$H = npzz^* + R, \tag{25}$$

- where the random matrix

$$R = H - \mathbb{E}H \tag{26}$$

has elements with zero mean, with $R_{ii} = 0$, and for $i \neq j$

$$R_{ij} = \begin{cases} (1-p)e^{\imath(\theta_i - \theta_j)} & \text{with prob. } p \\ e^{\imath\varphi} - pe^{\imath(\theta_i - \theta_j)} & \text{w.p. } 1-p \text{ and } \varphi \sim Uniform([0, 2\pi)) \end{cases}. \tag{27}$$

- the variance of R_{ij} is

$$\mathbb{E}|R_{ij}|^2 = (1-p)^2 p + (1+p^2)(1-p) = 1 - p^2 \tag{28}$$

for $i \neq j$, and 0 for the diagonal elements.
- for $p = 1$: the variance is zero as all edges are good

Analysis of the complete graph angular synchronization (iii)

- distribution of the eigenvalues of the random matrix R follows Wigner's semi-circle law
- has support $[-2\sqrt{n(1-p^2)}, 2\sqrt{n(1-p^2)}]$
- largest eigenvalue $\lambda_1(R)$:
  - is concentrated near the right edge of the support
  - the universality of the edge of the spectrum implies that it follows the Tracy-Widom distribution even when the entries of R are non-Gaussian
- leads to the approximation

$$\lambda_1(R) \approx 2\sqrt{n(1-p^2)} \tag{29}$$

Analysis of the complete graph angular synchronization (iv)

- matrix $H = npzz^* + R$ can be construed as a rank-one perturbation to a random matrix
- the distribution of its largest eigenvalue studied in the literature; Feral & Peche (2007) showed that if

$$np > \sqrt{n(1 - p^2)} \tag{30}$$

- then the largest eigenvalue $\lambda_1(H)$ will jump outside the support of the semi-circle law, and
- $\lambda_1(H)$ is normally distributed with mean μ and variance σ^2

$$\lambda_1(H) \sim \mathcal{N}(\mu, \sigma^2), \tag{31}$$

$$\mu = \frac{np}{\sqrt{1 - p^2}} + \frac{\sqrt{1 - p^2}}{p}, \qquad \sigma^2 = \frac{(n+1)p^2 - 1}{np^2}(1 - p^2) \tag{32}$$

Analysis of the complete graph angular synchronization (v)

- with some extra work: can lower bound the correlation between the normalized top eigenvector v_1 of H and the ground truth signal vector z

$$|\langle z, v_1 \rangle|^2 \geq \frac{\lambda_1(H) - \lambda_1(R)}{np}, \tag{33}$$

- since the variance of the correlation of two random unit vectors in $\mathbb{R}^n$ is $1/n$, we obtain above random correlation values with ground truth whenever

$$\frac{\lambda_1(H) - \lambda_1(R)}{np} > \frac{1}{n} \tag{34}$$

- which translates to

$$p > \frac{1}{\sqrt{n}} \tag{35}$$

- as soon as $p > \frac{1}{\sqrt{n}}$, we should obtain above random correlations between the vector of angles z and the top eigenvector v_1 of H!

Semidefinite Programming relaxation

- Recall from the spectral relaxation derivation:

$$\underset{||z||^2=n}{\text{maximize}}\; z^* H z \tag{36}$$

and note: $z^* H z = \mathrm{Tr}(z^* H z) = \mathrm{Tr}(H\, z z^*) = \mathrm{Tr}(H \Upsilon)$

$$\sum_{i,j=1}^{n} e^{-\iota\theta_i} H_{ij} e^{\iota\theta_j} = \mathrm{Tr}(H\Upsilon), \tag{37}$$

  - Υ is the (unknown) $n \times n$ Hermitian matrix of rank-1

$$\Upsilon_{ij} = e^{\iota(\theta_i - \theta_j)} \tag{38}$$

with ones in the diagonal $\Upsilon_{ii}, \forall i = 1, 2, \ldots, n$.

- Dropping the rank-1 constraint on Υ

$$\begin{aligned} \underset{\Upsilon \in \mathbb{C}^{n\times n}}{\text{maximize}} \quad & \mathrm{Tr}(H\Upsilon) \\ \text{subject to} \quad & \Upsilon_{ii} = 1 \qquad i = 1, \ldots, n \\ & \Upsilon \succeq 0, \end{aligned} \tag{39}$$

  - the recovered solution is not necessarily of rank-1
  - estimator obtained from the best rank-1 approximation

The Group Synchronization Problem over SO(d)

- finding group elements from noisy measurements of their ratios
- synchronization over $SO(d)$ consists of estimating a set of n unknown $d \times d$ matrices $R_1, \ldots, R_n \in SO(d)$ from a noisy measurements of a small subset of the pairwise ratios $Q_{ij} = R_i R_j^{-1} \in SO(d), \;\; (ij) \in G$

$$\underset{R_1,\ldots,R_n \in SO(d)}{\text{minimize}} \sum_{(i,j) \in E} w_{ij} || R_i^{-1} R_j - Q_{ij} ||_F^2 \tag{40}$$

- w_{ij} are non-negative weights representing the confidence in the available noisy pairwise measurements Q_{ij}
- the graph of available measurements if denoted as the **measurement graph** G

The Graph Realization Problem

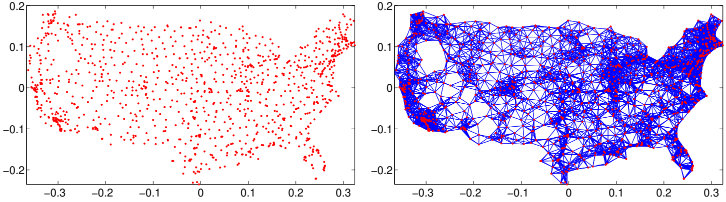


Figure: Original US map with $n = 1090$ and the measurement graph with sensing radius $\rho = 0.032$.

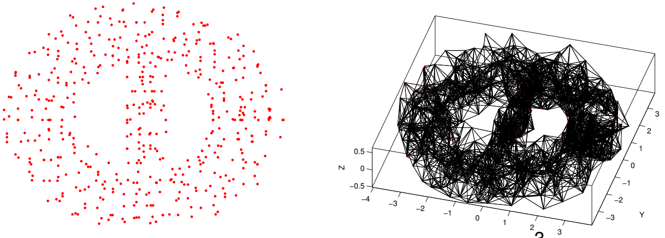


Figure: BRIDGE-DONUT data set of $n = 500$ points in $\mathbb{R}^3$ and the measurement graph of radius $\rho = 0.92$.

The Graph Realization Problem in $\mathbb{R}^d$

- Graph $G = (V, E)$, $|V| = n$ nodes
- Set of distances $l_{ij} = l_{ji} \in \mathbb{R}$ for every pair $(i, j) \in E$
- Goal: find a d-dimensional embedding $p_1, \ldots p_n \in \mathbb{R}^d$ s.t.

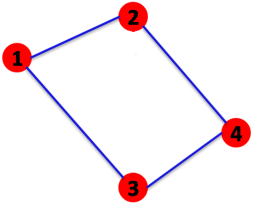
$$||p_i - p_j|| = l_{ij}, \text{ for all } (i, j) \in E$$

- If the solution is unique (up to a rigid motion), then graph is **globally rigid** (uniquely realizable)
- Noise $d_{ij} = l_{ij}(1 + \epsilon_{ij})$ where $\epsilon_{ij} \sim Uniform([-\eta, \eta])$
- Disc graph model with sensing radius ρ, $d_{ij} \leq \rho$ iff $(i, j) \in E$

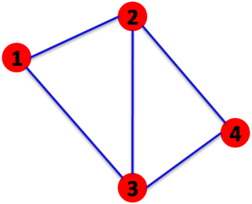
Practical applications:

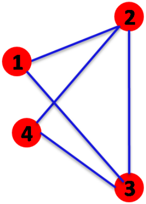
- Input: sparse noise subset of pairwise distances between sensors/atoms
- Output: d-dimensional coordinates of sensors/atoms

Local and Global Rigidity

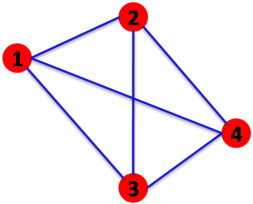


Flexible
(not locally rigid)





Locally rigid
(not globally rigid)



Globally rigid

Divide and conquer: a useful paradigm

1. Break the original measurement graph into many overlapping subgraphs (”patches”)
2. Embed all patches using one of the existing methods
3. Integrate all local embeddings in a global solution

Motivation:

- solvers are too slow for large graphs and not very accurate
- locally, the small subgraphs are dense, and can be embedded more robustly (and faster)

Breaking up the large graph into patches

- Find maximal globally rigid components in the 1-hop neighborhood graph (look for 3-connected components)

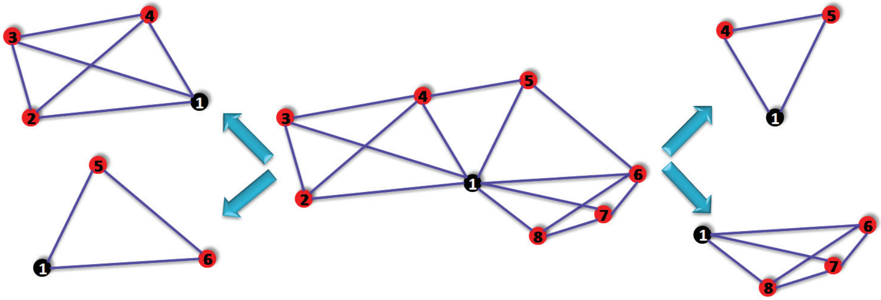


Figure: The neighborhood graph of center node 1 is split into four maximally 3-connected-components (patches):
$\{1,2,3,4\},\{1,4,5\},\{1,5,6\},\{1,6,7,8\}$.

Pairwise alignment of patches

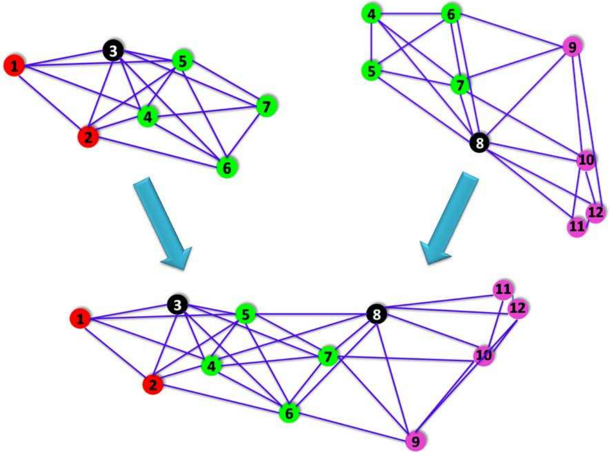
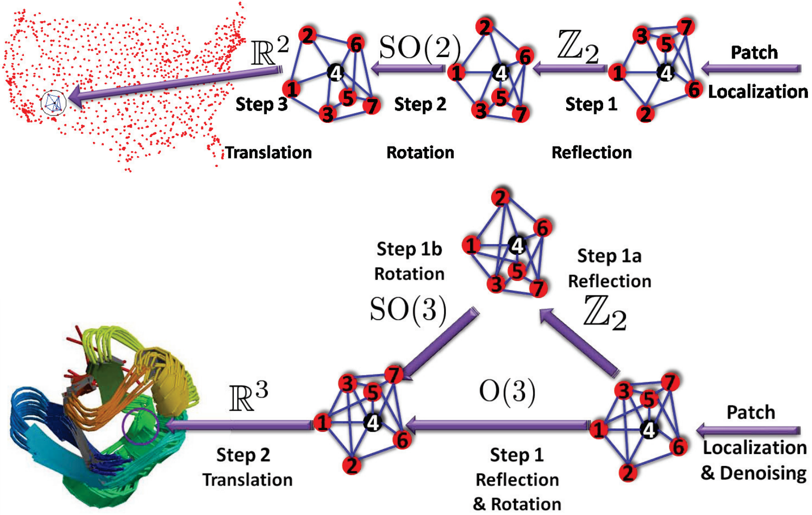


Figure: Optimal alignment of two patches that overlap in four nodes (provides a measurement for the ratio of the two group elements in Euc(2)).

Local frames and synchronization in $\mathbb{R}^d$



The rightmost subgraph is the embedding of the patch in its own local frame (stress minimization or SDP).

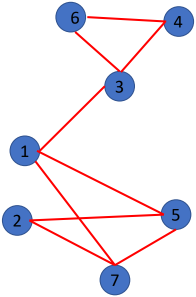
Synchronization - solving a big puzzle

Figure: To each piece P_i of the puzzle, we need to associate a certain translation t_i and rotation O_i (ok, here there are no reflections, or they are easy to handle), such that when we apply this set of transformations to each individual piece, everything ”clicks/synchronizes” together.

Source: https://www.vox.com/2020/4/8/21204424/jigsaw-puzzles-self-care-popularity

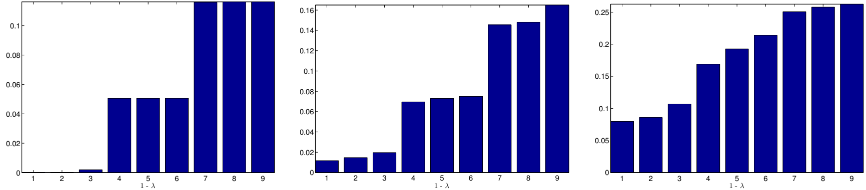
Measurement graph of pairwise ratios of group elements

	R_1	R_2	R_3	R_4	R_5	R_6	R_7
R_1	I_{dxd}		$R_1 R_2^{-1}$		$R_1 R_5^{-1}$		$R_1 R_7^{-1}$
R_2		I_{dxd}			$R_2 R_5^{-1}$		$R_2 R_7^{-1}$
R_3			I_{dxd}	$R_3 R_4^{-1}$		$R_3 R_6^{-1}$	
R_4				I_{dxd}		$R_4 R_6^{-1}$	
R_5					I_{dxd}		$R_5 R_7^{-1}$
R_6						I_{dxd}	
R_7							I_{dxd}



Synchronization over O(d) with noise

$$H_{ij} = \begin{cases} R_{ij} & (i,j) \in E \text{ (available group ratio measurement)} \\ O_{3\times 3} & (i,j) \notin E \text{ (no measurement available)} \end{cases}$$



(a) $\eta = 0\%$, $\tau = 0\%$, and $MSE = 6e-4$ (b) $\eta = 20\%$, $\tau = 0\%$, and $MSE = 0.05$ (c) $\eta = 40\%$, $\tau = 4\%$, and $MSE = 0.36$

Figure: Bar-plot of the top 9 eigenvalues of $\mathcal{H} = D^{-1}H$ for the UNITCUBE and various noise levels η. Note that we plot $1 - \lambda^{\mathcal{H}}$.

The Group Synchronization Problem

- finding group elements from noisy measurements of their ratios
- synchronization over $SO(d)$ consists of estimating a set of n unknown $d \times d$ matrices $R_1, \ldots, R_n \in SO(d)$ from a noisy measurements of a small subset of the pairwise ratios $Q_{ij} = R_i R_j^{-1} \in SO(d), \quad (ij) \in G$

$$\underset{R_1,\ldots,R_n \in SO(d)}{\text{minimize}} \sum_{(i,j)\in E} w_{ij} \| R_i^{-1} R_j - Q_{ij} \|_F^2 \tag{41}$$

- w_{ij} are non-negative weights representing the confidence in the available noisy pairwise measurements Q_{ij}
- the graph of available measurements if denoted as the **measurement graph** G

SDP and Spectral Relaxations

- Least squares solution to synchronization over $R_1, \ldots, R_n \in SO(d)$ that minimizes

$$\underset{R_1,\ldots,R_n \in SO(d)}{\text{minimize}} \sum_{(i,j)\in E} w_{ij} ||R_i^{-1} R_j - R_{ij}||_F^2 \tag{42}$$

$$\underset{R_1,\ldots,R_n \in SO(d)}{\text{maximize}} \sum_{(i,j)\in E} w_{ij} tr(R_i^{-1} R_j R_{ij}^T) \tag{43}$$

Rewrite objective as tr(G C)

- with $G_{ij} = R_i^T R_j$, and

$$G = \overline{R}^T \overline{R}$$

- where $\overline{R}_{d\times nd} = [R_1 R_2 \ldots R_n]$
- G is **unknown**

$$C_{ij} = w_{ij} R_{ij}^T (w_{ji} = w_{ij}, R_{ji} = R_{ij}^T)$$

- C is **known**

SDP and Spectral Relaxations SO(d)

- SDP Relaxation (Singer, 2011)

$$\begin{aligned} \underset{G}{\text{maximize}} \quad & tr(GC) \\ \text{subject to} \quad & G \succeq 0 \\ & G_{ii} = I_d, \text{for } i = 1, \dots n \\ & [\text{rank}(G) = d] \\ & [\det(G_{ij}) = 1, \text{for } i,j = 1, \dots n] \end{aligned} \tag{44}$$

- Spectral Relaxation: via the graph Connection Laplacian L

Let $C \in \mathbb{R}^{nd \times nd}$ with blocks $C_{ij} = w_{ij} R_{ij}$

Let $D \in \mathbb{R}^{nd \times nd}$ diagonal with $D_{ii} = d_i I_d$ where $d_i = \sum_j w_{ij}$

$$L = D - C, \qquad \text{with} \quad L\overline{R}^T = 0$$

- recover the rotations from the bottom d eigenvectors of L
- followed by SVD for rounding in the noisy case.