Part A Simulation and Statistical Programming HT19

Problem Sheet 1 – due Week 2 Thursday 10am 24-29 St Giles

1. Consider the integral
$$\theta = \int_0^\pi x \cos x dx.$$
   (a) Evaluate θ.
   (b) Give a Monte Carlo estimator $\widehat{\theta}_n$ for numerically approximating θ, using uniform random variables on $[0, \pi]$.
   (c) Calculate the bias and the variance of this estimator.
   (d) Using Chebyshev's inequality, determine how large n needs to be to ensure that the absolute error between $\widehat{\theta}_n$ and θ is less than 10^{-3}, with probability exceeding 0.99.
   (e) Same question using the Central Limit Theorem.

2. Consider the family of distributions with probability density function (pdf)
$$f_{\mu,\lambda}(x) = \lambda \exp\left(-2\lambda |x - \mu|\right), \qquad x \in \mathbb{R},$$
   where $\lambda > 0$ and $\mu \in \mathbb{R}$ are parameters.
   (a) Given $U \sim \mathrm{U}[0,1]$, use the inversion method to simulate from $f_{\mu,\lambda}$.
   (b) Let X have pdf $f_{\mu,\lambda}$. Show that $a + bX$ has pdf $f_{\mu',\lambda'}$ for $b \neq 0$. Find the parameters μ', λ'.
   (c) Let $Y, Z \sim \mathrm{Exp}(r)$. Show that $Y - Z$ has pdf $f_{\mu',\lambda'}$. Find the parameters μ', λ'. Hence, use the transformation method to simulate from $f_{\mu,\lambda}$ for any $\lambda > 0$ and $\mu \in \mathbb{R}$, given $U_1, U_2 \sim \mathrm{U}[0,1]$ independent.

3. (a) Let $Y \sim \mathrm{Exp}(\lambda)$ and fix $a > 0$. Let $X = Y | Y \geq a$. That is, the random variable X is equal to Y conditioned on $Y \geq a$. Calculate $F_X(x)$ and $F_X^{-1}(u)$. Give an algorithm simulating X from $U \sim \mathrm{U}[0,1]$.
   (b) Let a and b be given, with $a < b$. Show that we can simulate $X = Y | a \leq Y \leq b$ from $U \sim \mathrm{U}[0,1]$ using
$$X = F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U),$$
   i.e., show that if X is given by the formula above, then $\Pr(X \leq x) = \Pr(Y \leq x | a \leq Y \leq b)$. Apply the formula to simulate an exponential rv conditioned to be greater than a.
   (c) Here is a very simple rejection algorithm simulating $X = Y | Y > a$ for $Y \sim \mathrm{Exp}(\lambda)$:

1 Let $Y \sim \text{Exp}(\lambda)$. Simulate $Y = y$.

2 If $Y > a$ then stop and return $X = y$, and otherwise, start again at 1.

Calculate the expected number of trials to the first acceptance. Why is the inversion method to be preferred over this rejection algorithm for $a \gg 1/\lambda$?