4 Transportation and Assignment Problems

4.1 The Transportation Problem

A commodity is produced at \( m \) factories or sources \( S_1, \ldots, S_m \) and is sold at \( n \) markets or destinations \( D_1, \ldots, D_n \). The annual output or supply available at source \( S_i \) is \( s_i \) units, the annual demand at destination \( D_j \) is \( d_j \) units, and the cost of transporting one unit from \( S_i \) to \( D_j \) is \( c_{ij} \). We wish to determine which sources \( S_i \) should supply which destinations \( D_j \) so as to minimise transportation costs. We assume that each \( s_i > 0 \), \( d_j > 0 \), \( c_{ij} \geq 0 \).

Let \( x_{ij} \geq 0 \) be the number of units to be sent from \( S_i \) to \( D_j \) per year. Then we are to minimise the transportation cost

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}
\]

subject to the constraints that the amount taken from source \( S_i \) must be at most the supply \( s_i \), that is

\[
\sum_{j=1}^{n} x_{ij} \leq s_i \quad \text{for } i = i, \ldots, m,
\]

and the amount taken to market \( D_j \) must be at least the demand \( d_j \), that is

\[
\sum_{i=1}^{m} x_{ij} \geq d_j \quad \text{for } j = 1, \ldots, n.
\]

If there is any feasible solution \( x_{ij} \) then

\[
\text{total supply} = \sum_i s_i \geq \sum_i \sum_j x_{ij} \geq \sum_j d_j = \text{total demand}.
\]

If the total supply is at least the total demand, then we can easily adjust the problem to ensure that we have equality here and thus also in each of the inequalities above. We may do this by introducing if necessary an extra market (a dump) with demand equal to the excess supply \( \sum_i s_i - \sum_j d_j \), and setting all transportation costs to the dump equal to 0. (If the total demand exceeds the total supply, we may introduce a mythical extra source to make up the shortfall, and assign costs which reflect penalties for undersupplying markets.)

We thus define the transportation (or Hitchcock) problem as the following LP, where the \( s_i > 0 \), \( d_j > 0 \), \( c_{ij} \geq 0 \) are given, with \( \sum_i s_i = \sum_j d_j \).

\[
\min \sum_{i,j} c_{ij} x_{ij}
\]

\((P)\) subject to

\[
\sum_j x_{ij} = s_i \quad \text{for each } i = 1, \ldots, m
\]

\[
\sum_i x_{ij} = d_j \quad \text{for each } j = 1, \ldots, n
\]

\( x_{ij} \geq 0 \) for each \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \).
Variable production costs at factories can be built into the transportation costs — see exercise 3.4. Also, seemingly unrelated problems may sometimes be cast into this form, for example the ‘caterer’s problem’ in exercise 3.1 or the production scheduling problem in exercise 3.2.

The above LP may be written as

$$\min \mathbf{c}^\prime \mathbf{x} \quad \text{subject to} \quad A \mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0},$$

where for example in the case $m = 2$, $n = 3$

$$\mathbf{c} = (c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23})^\prime$$

$$\mathbf{x} = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})^\prime$$

$$\mathbf{b} = (-s_1, -s_2, d_1, d_2, d_3)^\prime$$

and

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

(The minus signs here are just for convenience below, when we consider the dual.)

**Dual program**

The variable $x_{ij}$ appears with coefficient -1 in the constraint for source $S_i$, and with coefficient +1 in the constraint for market $D_j$, and appears in no other constraint. Let $u_i$ and $v_j$ be respectively the dual variables corresponding to the $S_i$ and $D_j$ constraints. Then the dual constraint corresponding to the primal variable $x_{ij} \geq 0$ is

$$v_j - u_i \leq c_{ij}.$$

The dual program is thus

$$\max \sum_j d_jv_j - \sum_i s_iu_i$$

(D) subject to

$$v_j - u_i \leq c_{ij} \quad \text{for each } i, j$$

$$u_i, v_j \text{ unrestricted in sign.}$$

**An interpretation of the dual program**

A haulage contractor wants the transportation contract. He proposes to set local prices $u_i$, $v_j$ at each point $S_i$ or $D_j$. Thus he will buy at source $S_i$ at a unit price of $u_i$ and he will sell at market $D_j$ at a unit price of $v_j$, so that his effective charge for carrying one unit from $S_i$ to $D_j$ is $v_j - u_i$. The dual objective function is his net revenue, which he will try to maximise, and the dual constraints ensure that his effective charges are competitive.

Note that all the local prices $u_i$, $v_j$ may be made $> 0$ if desired by adding a large constant to each. This keeps them feasible, and does not change the objective function value, since the total supply equals the total demand.

**Example 1**

Suppose that we have two sources $S_1$ and $S_2$ with supplies $s_1 = 3$ and $s_2 = 5$, and three markets $D_1$, $D_2$ and $D_3$ with demands $d_1 = 4$, $d_2 = 2$ and $d_3 = 2$. We must use
the sources to satisfy the demands as cheaply as possible, where the unit transportation costs are given by the matrix

\[
(c_{ij}) = \begin{pmatrix}
3 & 5 & 4 \\
1 & 2 & 3
\end{pmatrix}
\]

**Preliminary comments**

(i) A transportation problem always has an optimal solution. For we may easily find a feasible solution for the primal (see below), and setting each \(u_i = v_j = 0\) gives a feasible solution for the dual; and so by the duality theorem both programs have optimal solutions.

(ii) The \((m + n) \times mn\) constraint matrix \(A\) has rank \(m + n - 1\). For the sum of all the rows of \(A\) is \(0\), and so \(A\) has rank \(\leq m + n - 1\). Also it is not hard to see that after deleting any one row the remaining rows are linearly independent.

We may suppose that we have dropped one equation, say the first. We now have a system of \(m + n - 1\) independent equations, and any solution to them automatically satisfies the dropped equation. A basic solution to our system will specify \(m + n - 1\) basic variables. If we do drop the first equation from the primal, then the variable \(u_1\) should not appear in the dual, and this is equivalent to setting \(u_1 = 0\).

(iii) Corresponding to a basic feasible solution, we may draw a graph with nodes \(S_1, \ldots, S_m\) and \(D_1, \ldots, D_n\), and with an edge joining \(S_i\) and \(D_j\) wherever \(x_{ij}\) is basic, so that there are exactly \((m + n - 1)\) edges. In example 1, a basic feasible solution is \((x_{ij}) = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}\) with basic variables \(x_{11}, x_{21}, x_{22}, x_{23}\); and the corresponding graph is

![Graph](example_graph.png)

It may be shown that a basic feasible solution always corresponds to a ‘spanning tree’ as here, that is to a graph without cycles which connects all the nodes.

(iv) Recall that a linear program is said to be degenerate if, in some basic feasible solution, some basic variable equals 0. It turns out that a transportation problem is degenerate if and only if a partial sum of the supplies \(s_i\) equals a partial sum of the demands \(d_j\), so that the routes used could fall into two separate systems. Thus example 1 above is non-degenerate, and example 2 below is degenerate.

**Solving the transportation problem**

Our method is essentially a slick version of the (revised) simplex method, and is based on the complementary slackness theorem. This says here that feasible solutions \(x_{ij}\) for the primal \((P)\) and \(u_i, v_j\) for the dual \((D)\) are optimal if and only if

\[v_j - u_i = c_{ij}\quad\text{whenever } x_{ij} > 0.\]

We list four steps.

1. Find an initial basic feasible solution to the primal \((P)\) (specifying exactly \(m + n - 1\) basic variables).

2. Find temporary local prices \(u_i, v_j\) by solving the \(m + n - 1\) equations

\[v_j - u_i = c_{ij}\quad\text{for } x_{ij} \text{ basic.}\]

These determine the \((m + n)\) prices \(u_i, v_j\) uniquely, provided that we set \(u_1 = 0\). [The \(u_i, v_j\) are the simplex multipliers, and the reduced cost for \(x_{ij}\) is \(\bar{c}_{ij} = c_{ij} + u_i - v_j\).]
(3) If the local prices $u_i, v_j$ also satisfy
\[ v_j - u_i \leq c_{ij} \]
then they are feasible for the dual problem $(D)$, and so by complementary slackness our solutions are optimal.

(4) If $v_j - u_i > c_{ij}$ for some non-basic $x_{ij}$ (so that the currently unused route from $S_i$ to $D_j$ looks attractive, with reduced cost $\bar{c}_{ij} = c_{ij} + u_i - v_j < 0$), then introduce $x_{ij}$ into the basis, dropping exactly one current basic variable, and return to step (2).

Step (1) is explained below. Steps (2), (3), (4) are best explained by examples.

In order to find an initial basic feasible solution in step (1), we first focus on a particular route (for example the cheapest route available), and allocate as much flow along it as possible. Then reduce the corresponding supply and demand appropriately, and 'kill off' the one reduced to zero. If two are reduced to zero simultaneously we kill off just one, except that we stop after introducing the $(m + n - 1)$st basic variable (which must reduce to zero the last active supply and demand). When we always choose a cheapest available route this is called the 'matrix minimum' method.

**Example 1 continued** (non-degenerate)

We use the matrix minimum method to get started. First we set

\[ x_{21} = \min(s_2, d_1) = \min(5, 4) = 4, \]

reduce $s_2$ to 1 and kill off $D_1$ leaving a $2 \times 2$ active tableau.

\[
\begin{array}{ccc}
3 & 5 & 4 \\
1 & 4 & 2 \\
\end{array}
\]

\[
\begin{array}{c}
s_1 = 3 \\
s_2 = 1 \\
\end{array}
\]

\[
\begin{array}{c}
d_1 = X \\
d_2 = 2 \\
d_3 = 2 \\
\end{array}
\]

Next set $x_{22} = \min(1, 2) = 1$, kill off $S_2$ and reduce the demand for $D_2$ to 1. Then set $x_{13} = \min(2, 3) = 2$ reduce the supply for $S_1$ to 1 and kill off $D_3$. Our tableau now is

\[
\begin{array}{ccc}
3 & 5 & 4 \\
1 & 4 & 2 \\
\end{array}
\]

\[
\begin{array}{c}
\bar{d}_1 = X \\
\bar{d}_2 = 1 \\
\bar{d}_3 = X \\
\end{array}
\]

Finally we are forced to set $x_{12} = 1$ which automatically satisfies both supply and demand simultaneously. We now present our initial basic feasible solution (with cost 19).
Next we set \( u_1 = 0 \) and use the tableau to determine the rest of the temporary local prices \( u_i, v_j \). First we set \( v_2 = 5 \) (since \( x_{12} \) is basic and so \( v_2 - u_1 = c_{12} \)) and \( v_3 = 4 \) (since \( v_3 - u_1 = c_{13} \)). Then \( u_2 = 3 \) (since \( v_2 - u_2 = c_{22} \)) and finally \( v_1 = 4 \) (since \( v_1 - u_2 = c_{21} \)).

We know that \( v_j - u_i = c_{ij} \) whenever \( x_{ij} \) is basic, so that \( \bar{c}_{ij} = c_{ij} + u_i - v_j = 0 \). For each non-basic \( x_{ij} \), we compute \( \bar{c}_{ij} \). We find that \( \bar{c}_{11} = -1 < 0 \), so that the route \( S_1 \) to \( D_1 \) is attractive.

We follow step 4, and introduce \( x_{11} \) into the basis at level \( t \geq 0 \). There is always a unique cycle involving the entering variable \( x_{11} \) and some of the basic variables (but no other non-basic variable), and we can alternately add and subtract \( t \) around the cycle, so that all supplies and demands stay satisfied.

Note that the change in cost here is

\[
 t(c_{11} - c_{12} + c_{22} - c_{21}) = t((c_{11} - (v_2 - u_1)) + (v_2 - u_2) - (v_1 - u_2)) = t(\bar{c}_{11} - (v_1 - u_2)) = t\bar{c}_{11} = -t
\]

(as we should expect!). We make \( t \) as large as possible, whilst keeping all \( x_{ij} \geq 0 \). Thus here \( t = 1 \), and we obtain the new basic feasible solution below (in which \( x_{12} \) has left the basis). Also, we calculate the prices \( u_i, v_j \) as before, starting with \( u_1 = 0 \).

We now find that \( v_2 - u_1 \leq c_{12} \) and \( v_3 - u_2 \leq c_{23} \), and find that we have reached an optimal solution, with cost 18.

**Comments**

(i) Since here \( \bar{c}_{ij} > 0 \) for each non-basic \( x_{ij} \), the optimal solution is unique.

(ii) We met only a cycle of length four, but in larger problems we could meet longer cycles.

(iii) If all supplies \( s_i \) and demands \( d_j \) are integers, as above, then the solutions \( x_{ij} \) we generate will always be integers, which is handy if you are transporting cars.
Example 2

\[
\begin{array}{ccc}
2 & 3 & 1 \\
4 & 5 & 2 \\
\end{array}
\begin{array}{c}
s_1 = 5 \\
s_2 = 2 \\
\end{array}
\]

Note that the problem is degenerate, since for example \( s_1 = d_1 + d_2 \), and so there is a feasible solution such that the graph corresponding to the positive variables \( x_{ij} \) is as below.

\[
\begin{array}{ccc}
S_1 & D_1 \\
S_2 & D_2 \\
& D_3 \\
\end{array}
\]

We use the matrix minimum method to obtain an initial basic feasible solution, specifying exactly \( m + n - 1 = 4 \) basic variables. Firstly we set \( x_{13} = 2 \), reduce the first supply to 3 and kill off the last column. Then set \( x_{11} = \min\{3, 3\} = 3 \). Both the first supply and first demand are now reduced to zero, and we must kill off just one, say the first supply.

\[
\begin{array}{ccc}
2 & 3 & 1 \\
4 & 5 & 2 \\
\end{array}
\]

Next set \( x_{21} = \min\{2, 0\} = 0 \) and kill off the first column, and finally \( x_{22} = 2 \). We now have an initial basic feasible solution, with cost 18. We obtain the five local prices \( u_i, v_j \) as before, by setting \( u_1 = 0 \) and using the four equations corresponding to the basic variables (including \( x_{21} \)).

\[
\begin{array}{ccc}
2 & 3 & 1 \\
4 & 5 & 2 \\
\end{array}
\]

Since \( v_3 - u_2 = 3 > c_{23} \), we introduce \( x_{23} \) into the basis. We must, however, set \( t = 0 \): we ‘stall’, and simply swap \( x_{23} \) for \( x_{21} \) in the basis.
Now $v_2 - u_1 = 4 > c_{12}$ and so we introduce $x_{12}$ into the basis. This time we set $t = 2$ and find that both $x_{13}$ and $x_{22}$ drop to zero. We must drop exactly one from the basis, say $x_{13}$.

We find that we have reached an optimum solution, with cost 16.

**Sensitivity analysis**

Our optimal solution in example 1 does not use $x_{12}$. What if the transportation cost $c_{12}$ were reduced? Note that at present $\bar{c}_{12} = c_{12} + u_1 - v_1 = 1$. If we reduce $c_{12}$ to $c_{12} - p$ then clearly the current optimal solution remains optimal if $0 \leq p \leq 1$, but not if $p > 1$.

What if the cost of a route currently used increases, say $c_{21}$ increases to $c_{21} + p$?

The current solution remains optimal if $4 - p - 0 \leq 5$ and $4 - (2 - p) \leq 3$, that is if $0 \leq p \leq 1$, but not if $p > 1$.

We have seen above that it is straightforward to deal with changes in transportation costs $c_{ij}$. Now consider changes in supplies and demands. The local prices $u_i$, $v_j$ will guide us about resulting changes in the total transportation costs.

Recall that the dual objective function is $\sum_j d_j v_j - \sum_i s_i u_i$. Thus if some $s_i$ and some $d_j$ are increased by 1 then the total cost will increase by $v_j - u_i$ (which could be negative), as long as the current local prices $u_i$, $v_j$ remain an optimal dual solution, and this will happen as long as we can still use the same basic variables (routes) in the primal problem.

Suppose in example 1 that both $s_2$ and $d_3$ are increased by 1. Then the total cost will increase by $v_3 - u_2$ if we can still use the same basic routes. But indeed we can do this, for we may set $x_{23} = 1$ and then ‘pivot it out’ by alternately adding and subtracting 1 around the cycle formed by $x_{23}$ and the basic variables. We obtain the (degenerate) optimal solution

which uses the same basic variables. Note that this procedure worked since the old optimal primal solution was non-degenerate, and so each basic variable had value at least 1. If
say \( s_1 \) and \( d_3 \) are both increased by 1, then the cost will increase by \( c_{13} = v_3 - u_1 = 4 \), since \( x_{13} \) is currently basic.

Now suppose that just the capacity limit \( s_2 \) for source \( S_2 \) is increased by 1, so that we have excess supply. It appears that we should leave the excess unit of supply at the source with the smallest local price \( u_i \) (which is source \( S_1 \) here), and then the overall cost should decrease by \( u_2 - u_1 \). We may check this by considering the old optimal tableau with an added dump with demand 1.

\[
\begin{array}{cccc}
3 & 1 & 5 & 4 & 2 & 0 \\
0 & 1 & 3 & 2 & 2 & 3 & 0 & 1 \\
\end{array}
\]

We have set \( x_{14} = 0 \), \( x_{24} = 1 \) to obtain a basic feasible solution to the new problem. Note that \( v_4 = u_2 \). Since \( u_1 < u_2 \) we have \( v_4 - u_1 > 0 \). We pivot \( x_{14} \) into the basis, and we choose to drop \( x_{24} \).

\[
\begin{array}{cccc}
3 & 0 & 5 & 4 & 2 & 0 & 1 \\
0 & 1 & 4 & 2 & 2 & 3 & 0 \\
\end{array}
\]

The only local price that changes is \( v_4 \), which now equals \( u_1 \). The current solution is optimal since \( u_1 \) is the smallest of the costs \( u_i \).

### 4.2 The assignment problem

Suppose that we are to assign \( n \) jobs to \( n \) machines, one per machine. When job \( i \) is assigned to machine \( j \) this incurs a cost \( c_{ij} \), and we wish to minimise the total cost. We may formulate this problem as the following integer linear program, by using \( \{0, 1\} \)-valued variables \( x_{ij} \), where \( x_{ij} = 1 \) corresponds to assigning job \( i \) to machine \( j \).

\[
\begin{align*}
\text{min} & \quad \sum_i \sum_j c_{ij} x_{ij} \\
\text{subject to} & \quad \sum_j x_{ij} = 1 \quad \text{for each } i = 1, \ldots, n \\
& \quad \sum_i x_{ij} = 1 \quad \text{for each } j = 1, \ldots, n \\
& \quad x_{ij} = 0 \text{ or } 1 \quad \text{for each } i, j = 1, \ldots, n.
\end{align*}
\]

**Comments** (i) If say there are more machines than jobs, we may add fictitious jobs with zero costs to obtain the above model. (ii) From our discussion of the transportation problem, we see that we may consider the LP ‘relaxation’ in which we have replaced \( x_{ij} = 0 \) or 1 above by \( x_{ij} \geq 0 \). We thus obtain an LP which always has an optimal solution in which each \( x_{ij} = 0 \) or 1. (iii) The LP relaxation and its dual always have optimal solutions. The dual may be written
max \( \sum_i u_i + \sum_j v_j \) subject to \( u_i + v_j \leq c_{ij} \) for each \( i, j \).

We could solve the problem by using the transportation algorithm described earlier. However, our problem now is highly degenerate: each basic feasible solution has \( n \) basic variables equal to 1 and thus has \( n - 1 \) equal to 0. We sketch below a ‘primal-dual’ approach, which is based on three simple observations.

- Adding a constant \( \gamma \) to any row or column of the cost matrix \( (c_{ij}) \) does not change the optimal solutions, since the cost of each feasible solution changes by exactly \( \gamma \).
- Thus we can ensure that all costs are \( \geq 0 \), and we can introduce many zero costs.
- If we have an assignment using only zero costs it must be optimal.

**Example 1**

\[
(c_{ij}) = \begin{pmatrix} 5 & 7 & 9 \\ 14 & 10 & 12 \\ 15 & 13 & 16 \end{pmatrix}
\]

We subtract the row minimum \( u_i \) from each row \( i \), and then the net column minimum \( v_j \) from each column \( j \).

\[
\begin{array}{l|ccc}
 & v_1 = 0 & v_2 = 0 & v_3 = 2 \\
\hline
u_1 = 5 & 5 & 7 & 9 \\
u_2 = 10 & 14 & 10 & 12 \\
u_3 = 13 & 15 & 13 & 16 \\
\end{array}
\]

The resulting reduced costs \( \tilde{c}_{ij} = c_{ij} - u_i - v_j \) are given by

\[
(\tilde{c}_{ij}) = \begin{pmatrix} 0 & 2 & 2 \\ 4 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}
\]

The (unique) optimal solution for these costs is

\[
(x_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

and this is then the (unique) optimal solution for the original costs, with total cost 30.

The \( u_i, v_j \) always form a feasible solution to the dual problem \((D)\), since all reduced costs \( \tilde{c}_{ij} = c_{ij} - u_i - v_j \) are \( \geq 0 \). Indeed, they form an optimal dual solution here, by complementary slackness. (Check also that \( \sum_i u_i + \sum_j v_j = 30 \).)

By a complementary slackness theorem, there must always be some dual-feasible \( u_i, v_j \) such that there is an assignment just using zero net costs \( \tilde{c}_{ij} \). We were lucky above but...

**Example 2**

\[
(c_{ij}) = \begin{pmatrix} 5 & 2 & 3 & 4 \\ 7 & 8 & 4 & 5 \\ 6 & 3 & 5 & 6 \\ 2 & 2 & 3 & 5 \end{pmatrix}
\]

We proceed as before.
\[ u_1 = 2 \]
\[ u_2 = 4 \]
\[ u_3 = 3 \]
\[ u_4 = 2 \]
\[ v_1 = 0 \quad v_2 = 0 \quad v_3 = 0 \quad v_4 = 1 \]

We focus on the positions where the \( \bar{c}_{ij} \) are zero. We may spot a partial assignment using 3 zeros, shown in bold above. We may also spot 3 lines, for example row 2 and columns 1 and 2, which cover all the zeros. This shows that we can do no better, since we can pick at most one zero from any line.

König’s theorem states that, if we can assign only \( k \) zeros, then there are \( k \) lines which cover all the zeros. Indeed, there is a good method for finding a largest partial assignment, using say \( k \) zeros, and a corresponding cover with \( k \) lines. For small problems this step can be done by inspection.

Our next step is to improve the \( u_i, v_j \) by using the cover by lines that we have found. Let \( \delta \) be the minimum uncovered element in the \( \bar{c}_{ij} \) matrix, so that \( \delta > 0 \). Here \( \delta = 1 \). Subtract \( \delta \) from each uncovered row (to introduce zeros into the previously barren region) and add \( \delta \) to each covered column (to ensure that the new \( \bar{c}_{ij} \) are \( \geq 0 \)). Note that this is the same as subtracting \( \delta \) from each uncovered element and adding \( \delta \) to each doubly covered element. We obtain

\[
(\bar{c}_{ij}) = \begin{pmatrix}
3 & 0 & 1 & 1 \\
3 & 4 & 0 & 0 \\
3 & 0 & 2 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix}
\]