Analyzing the Joint Dynamics of Several Networks

Tom A.B. Snijders

University of Oxford
University of Groningen

June 2025

1. Multiple Networks — ???

Social actors are embedded in multiple networks

1. Multiple Networks — ???

Social actors are embedded in multiple networks

friendship, getting along well, admiration, advice, ...

1. Multiple Networks — ???

Social actors are embedded in multiple networks

friendship, getting along well, admiration, advice, ...

friendship, bullying, defending, dislike, ...

1. Multiple Networks — ???

Social actors are embedded in multiple networks

friendship, getting along well, admiration, advice, ...

friendship, bullying, defending, dislike, ...

friendship, esteem, collaboration, trust, advice, enmity, ...

1. Multiple Networks — ???

Social actors are embedded in multiple networks

friendship, getting along well, admiration, advice, ...

friendship, bullying, defending, dislike, ...

friendship, esteem, collaboration, trust, advice, enmity, ...

collaborative projects, client referral, information sharing, ...

But when analyzing networks, attention is often focused on ‘*the*’ network, as if there is only one.

A *multiple* or *multivariate* social network is a set of n social actors, on which $R \geq 2$ relations are defined (Wasserman & Faust, 1994; Pattison & Wasserman, 1999).

But when analyzing networks, attention is often focused on '*the*' network, as if there is only one.

A *multiple* or *multivariate* social network is a set of n social actors, on which $R \geq 2$ relations are defined (Wasserman & Faust, 1994; Pattison & Wasserman, 1999).

The study of multiple networks is quite traditional:
e.g., White, Boorman & Breiger (1976);
Boorman & White (1976); Pattison (1993);
later on, authors including Ibarra, Krackhardt, Padgett, Lazega, Lomi, did empirical research on multiple networks.

A term often used is *multiplexity*,
which draws attention to the <u>number</u> of different relations
that may exist between two actors.

E.g., two children play with each other,
are members of the same sports club,
are in the same classroom $\Rightarrow$ multiplexity = 3.

But the qualitative differences are more important
than just the number!

The term ‘multiplexity’ or ‘multiplex networks’ also is used often
with the same meaning as ‘multivariate’.

Another term is ‘multilayer’.

2. Choice of networks in a multivariate approach

When following a multivariate network approach,
the first step is to choose the network dimensions under study.

Parsimony $\Leftrightarrow$ *Completeness*

2 or 3 relations is manageable,
from 4 on it starts getting more and more complicated.

2. Choice of networks in a multivariate approach

When following a multivariate network approach,
the first step is to choose the network dimensions under study.

Parsimony $\Leftrightarrow$ *Completeness*

2 or 3 relations is manageable,
from 4 on it starts getting more and more complicated.

What are the few crucial network dimensions?

2. Choice of networks in a multivariate approach

When following a multivariate network approach,
the first step is to choose the network dimensions under study.

Parsimony ⇔ *Completeness*

2 or 3 relations is manageable,
from 4 on it starts getting more and more complicated.

What are the few crucial network dimensions?

This totally depends on the research domain & questions,
and has to be argued theoretically.

... some examples ...

For organizational research, Emmanuel Lazega
('The Collegial Phenomenon') proposes as crucial dimensions:
Friendship – Collaboration – Advice.

In studies of Dislike or other 'negative' relations,
these are often combined with Friendship.

For studying bullying in schools,
the combination Friendship – Bullying – Defending is important
for showing 'the nature' of bullying.

For corruption networks, a relevant set of networks may be
Profit giving – Authority – Informal embeddedness.

... some examples ...

For organizational research, Emmanuel Lazega
('The Collegial Phenomenon') proposes as crucial dimensions:
Friendship – Collaboration – Advice.

In studies of Dislike or other 'negative' relations,
these are often combined with Friendship.

For studying bullying in schools,
the combination Friendship – Bullying – Defending is important
for showing 'the nature' of bullying.

For corruption networks, a relevant set of networks may be
Profit giving – Authority – Informal embeddedness.

In addition, think of two-mode (affiliation) networks!

General principles for choice of relations when trying to explain networks

Some tentative general principles for the choice of a small number of crucial network dimensions in a given domain:

- The baseline of contact opportunities sometimes is given by the network delineation (e.g., classroom).
  Especially in larger groups (≥ 30), additional contextual indications may be used to represent contact opportunities.
  These can be exogenous (e.g., classroom membership) or endogenous (e.g., acquaintance, club membership);
  this can then be used in statistical models,
  e.g., as a dyadic covariate,
  or as a monadic covariate with the `sameX` effect,
  or as a secondary dependent network variable.

- More specific networks, e.g., negative networks or advice networks, can be better understood when studying them against the background of more general positive association networks, such as friendship.
- Joint activities may be an important dimension.
  These can often be represented as two-mode networks.
  E.g., bars frequented; club membership; courses taken.
  This can also be used to represent contact opportunities.
- Two-mode networks representing the possibility of joint activity
  (e.g., sport or other club membership)
  have a different nature than two-mode networks representing characteristics
  (e.g., subscribing to attitudinal statements, participating in sports).

3. Relations between relations

... on the variety of how relations can affect relations ...

3. Relations between relations

... on the variety of how relations can affect relations ...

(cf. also the algebraic approach; e.g., work by Pattison & Breiger.)
e.g.

$$P^2 = G$$

meaning that a parent of a parent is a grandparent.

However, here we treat a longitudinal statistical approach.

3. Relations between relations

... on the variety of how relations can affect relations ...

(cf. also the algebraic approach; e.g., work by Pattison & Breiger.)
e.g.

$$P^2 = G$$

meaning that a parent of a parent is a grandparent.

However, here we treat a longitudinal statistical approach.

It's a multilevel issue (but not nested):

ties, dyads, actors, triads, subgroups, ...

Different relations can impinge on one another in many different ways.

Example: friendship $\Rightarrow$ advice asking; ego is $\otimes$.

In the first place, within-dyad.

direct association (within tie)
‘friends become ’

Different relations can impinge on one another in many different ways.

Example: friendship $\Rightarrow$ advice asking; ego is $\otimes$.

In the first place, within-dyad.

direct association (within tie)
'friends become advisors'

Different relations can impinge on one another in many different ways.

Example: friendship $\Rightarrow$ advice asking; ego is $\otimes$.

In the first place, within-dyad.

direct association (within tie)
‘friends become advisors’

mixed reciprocity
‘friendship reciprocated

Different relations can impinge on one another in many different ways.

Example: friendship ⇒ advice asking; ego is ⊗.

In the first place, within-dyad.

direct association (within tie)
‘friends become advisors’

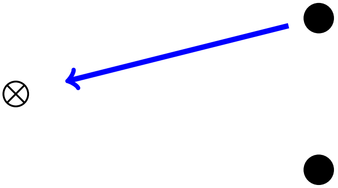
mixed reciprocity
‘friendship reciprocated
by asking advice’

A second category operates via actors.

mixed popularity

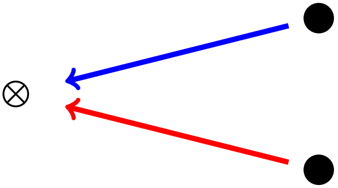
‘those popular as friends

’



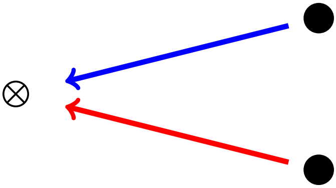
A second category operates via actors.

mixed popularity
‘those popular as friends
are asked a lot for advice’

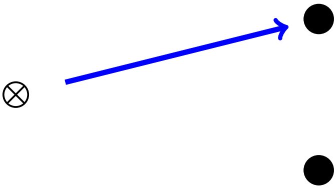


A second category operates via actors.

mixed popularity
'those popular as friends
are asked a lot for advice'

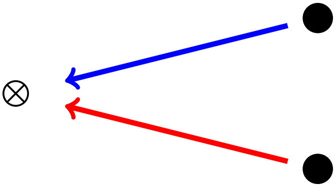


mixed activity
'those mentioning many friends
'

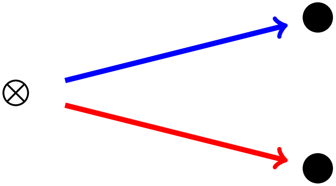


A second category operates via actors.

mixed popularity
'those popular as friends
are asked a lot for advice'

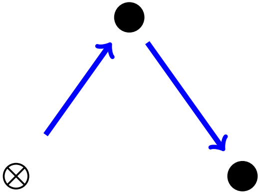


mixed activity
'those mentioning many friends
also mention many advisors'



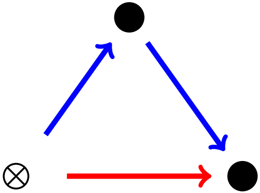
Next category: triads.

mixed transitive closure
‘friends of friends



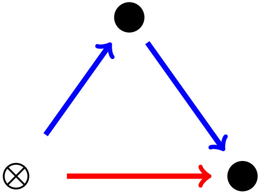
Next category: triads.

mixed transitive closure
‘friends of friends
become advisors’

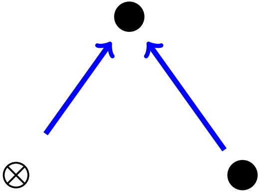


Next category: triads.

mixed transitive closure
‘friends of friends
become advisors’

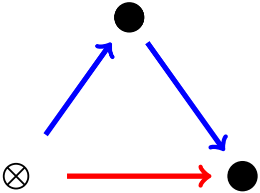


agreement
‘those with the same friends

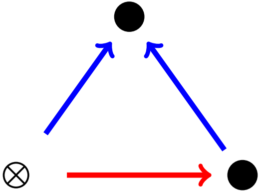


Next category: triads.

mixed transitive closure
'friends of friends
become advisors'

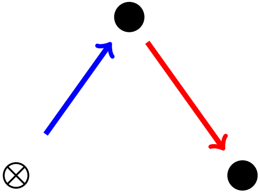


agreement
'those with the same friends
become advisors'



More triads.

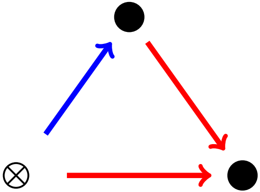
other mixed transitive closure
‘advisors of friends



Actor orientation: only the bottom tie is the dependent variable.

More triads.

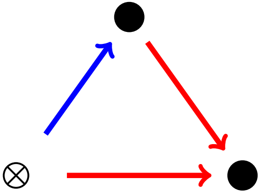
other mixed transitive closure
'advisors of friends
become advisors'



Actor orientation: only the bottom tie is the dependent variable.

More triads.

other mixed transitive closure
'advisors of friends
become advisors'



Actor orientation: only the bottom tie is the dependent variable.

And there are more mixed triads.

This type of cross-network dependencies is discussed for cross-sectional observations in Wasserman & Pattison (1999), (ERGMs) with examples in Lazega & Pattison (1999).

For longitudinal observations the dependencies are multiplied, because we must distinguish between the dependent and the explanatory (antecedent – subsequent) relations.

This can also be applied to *signed graphs* in which case balance theory can be applied.

4. Co-evolution of Multiple Networks

Like the basic Stochastic Actor-oriented Model, the model is a model for change of networks in which evolution in continuous time is assumed; the 'state' of the process now is the combination of the several networks.

each dependent network $X^{[r]}$ has its own rate function $\lambda^{[r]}$
and its own objective function $f_i^{[r]}$,
depending on all networks,
which leads to their mutual dependence / entwinement
in a joint feedback process.

Network modeling emphasizes the representation of structure,
which corresponds to the statistical dependence of tie variables.

For multivariate networks, the internal structure of the network is a given,
and is extended with the interdependence of the multiple networks.

In the actor-oriented approach, this dependence is organized by ego,
the actor who supposedly chooses changes in outgoing ties;
in this case, outgoing ties in several networks.

Choices are organized in such a way that
potential ties in the <u>same</u> network are compared,
but there is no comparison between ties in <u>different</u> networks.

In other words, the option sets of choices
always consist of tie variables in <u>one</u> network;
but past choices in the one network influence later choices in other networks.

Outline of the co-evolution model: mini-step

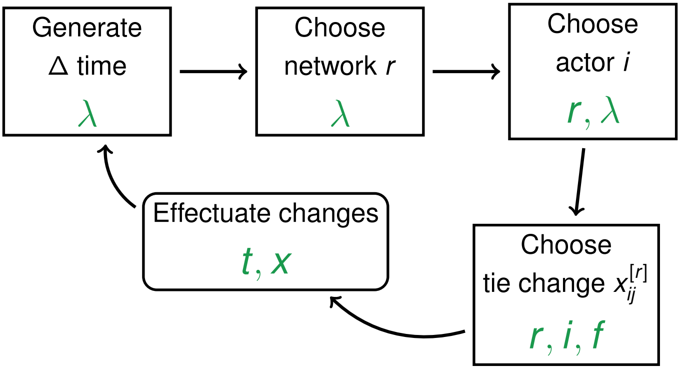
Suppose there are networks $X^{[1]}, X^{[2]}, \ldots, X^{[R]}$, for some number $R \geq 2$.

Their co-evolution proceeds in the following *smallest* steps:

1. at a random 'next' moment, an actor i is chosen,
   and a network $X^{[r]}$ is selected with $1 \leq r \leq R$;
2. actor i chooses an actor j for creating
   or dropping the tie $i \stackrel{r}{\to} j$ in network $X^{[r]}$,
   or leaves everything unchanged;
   choice probabilities depend potentially on <u>all</u> networks;
3. the change (if any) is put into effect, and the process restarts.

Flow chart for the mini-step

The co-evolution Markov chain is a succession of mini-steps:



Specification: Multiple networks require multilevel thinking

Interdependencies between networks can play on various levels;
e.g., for friendship and advice:

1. dyadic entrainment: friends become advisors;
2. dyadic exchange:
   I ask advice from those who say I am their friend;
3. actor level: those who have many friends get many advisors
   (not necessarily the same persons)
   (4 combinations in/outdegrees);
4. mixed closure 1: friends of friends become advisors;
5. mixed closure 2: advisors of friends become advisors;
6. and other mixed closures.

(See Snijders, Lomi, Torlò 2013; Snijders, 2016)

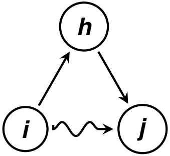
5. Effects in RSiena

The network in the role of dependent variable is called X and the network in the explanatory role is called W: `name = X`, `interaction1 = W`

	Description	*formula*	`shortName`
1	Direct entrainment	$\sum_j w_{ij}\, x_{ij}$	`crprod`
2	Reciprocal entrainment	$\sum_j w_{ji}\, x_{ij}$	`crprodRecip`
3	Mixed indegree-popularity	$\sum_j w_{+j}\, x_{ij}$	`inPopIntn`
4	Mixed indegree-activity	$\sum_j w_{+i}\, x_{ij}$	`inActIntn`
5	Mixed outdegree-popularity	$\sum_j w_{j+}\, x_{ij}$	`outPopIntn`
6	Mixed outdegree-activity	$\sum_j w_{+i}\, x_{ij}$	`outActIntn`
7	same outgoing ties	$\sum_{j \neq h} x_{ij}\, w_{ih}\, w_{jh}$	`from`
8	WXX closure	$\sum_{j \neq h} w_{ih}\, x_{ij}\, x_{hj}$	`to`
9	WWX closure	$\sum_{j \neq h} w_{ih}\, w_{hj}\, x_{ij}$	`closure`
10	XWX closure	$\sum_{j \neq h} x_{ij}\, w_{hj}\, x_{ih}$	`cl.XWX`

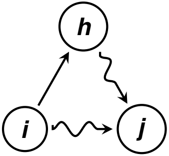
For the closure effects, see the next page.

Some triadic effects



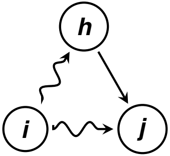
WWX closure

`closure`



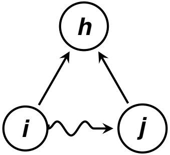
WXX closure

`to`



XWX closure

`cl.XWX`



same out-***W***-ties

`from`

The focal actor is *i*.

Dependent variable *X*: curly arrows;

explanatory variable *W*: straight arrows.

Names mentioned: first a somewhat understandable name,
then the RSiena `shortName`.

6. Example

Research with Vanina Torlo and Alessandro Lomi.

International MBA program in Italy;
75 students; 3 waves.

1. *Friendship*
2. *Advice*:
   To whom do you go for help if you missed a class, etc.
3. Important covariate: *performance*

Results 1: within-network & covariate effects

	Friendship		*Advice*	
Effect	par.	(s.e.)	par.	(s.e.)
outdegree (density)	–2.944***	(0.155)	–3.751***	(0.264)
reciprocity	1.605***	(0.252)	1.133***	(0.245)
transitive triplets	0.178***	(0.024)	0.210***	(0.053)
transitive recipr. triplets	–0.143***	(0.039)	0.027	(0.090)
indegree - popularity	0.037***	(0.010)	0.0443***	(0.0075)
outdegree - popularity	–0.029***	(0.007)	0.024	(0.027)
outdegree - activity	0.0071	(0.0082)	0.050***	(0.015)
reciprocated degree - activity	–0.007	(0.031)	–0.118**	(0.042)
gender alter	0.043	(0.071)	0.027	(0.097)
gender ego	–0.092	(0.073)	–0.202*	(0.094)
same gender	0.194**	(0.070)	0.048	(0.091)
same nationality	0.213**	(0.081)	0.358**	(0.121)
performance alter	–0.035†	(0.021)	0.139***	(0.033)
performance ego	–0.103***	(0.021)	–0.014	(0.031)
performance squared ego	–		0.043***	(0.010)
performance difference squared	–0.0189***	(0.0045)	–0.0272***	(0.0074)

† $p < 0.1$; * $p < 0.05$; ** $p < 0.01$; *** $p < 0.001$;

convergence t ratios all < 0.02; overall maximum convergence ratio 0.07.

Results 2: cross-network effects

Effect	*Friendship* par.	*Friendship* (s.e.)	*Advice* par.	*Advice* (s.e.)
advice	1.602***	(0.246)	–	
incoming advice	0.810***	(0.193)	–	
friendship	–		1.426***	(0.233)
incoming friendship	–		0.565**	(0.217)
mixed indegree popularity	–0.044**	(0.015)	–0.031*	(0.013)
mixed outdegree popularity	–0.066***	(0.017)	–0.0044	(0.0058)
mixed outdegree activity	–0.046*	(0.023)	–0.046***	(0.011)
WWX closure	0.049	(0.103)	0.035	(0.038)
WXX closure	0.094	(0.087)	0.052	(0.042)
XWX closure	0.062†	(0.036)	–0.034	(0.038)

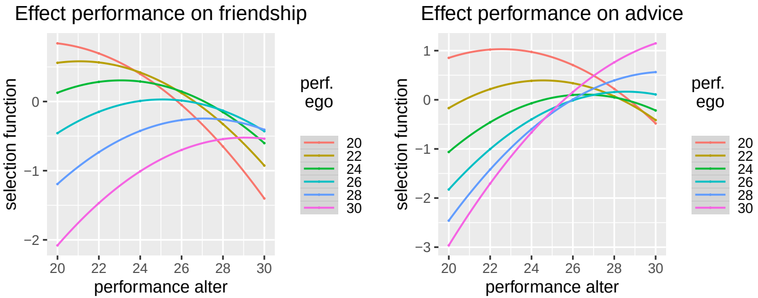
† $p < 0.1$; * $p < 0.05$; ** $p < 0.01$; *** $p < 0.001$;

convergence t ratios all < 0.02; overall maximum convergence ratio 0.07.

Selection functions

Not the focus of attention here.

See Snijders & Lomi, *Network Science*, 2019.



Co-evolution of friendship and advice: selection functions for performance.

Plots made by `SelectionTables.R` (see website).

It is interesting to compare multivariate (i.e., co-evolution) results
with univariate results for one network;
in the univariate results the network is considered as if it exists
independently of the other network/s.

Here the networks of sociability (friendship, getting along with)
may serve as the background context for the more specific networks (advice).

The following page repeats the results for the advice network
analysed co-evolving together with friendship,
and analysed on its own with, for the rest, the same model specification.

Here the function `updateSpecification` is handy;
if `TwoNetEffects` is the effects object with the multivariate specification:

```
AEFfects <- getEffects(AdviceData)
(AEFfects <- updateSpecification(AEFfects, TwoNetEffects))
```

Results 3: advice with and without friendship

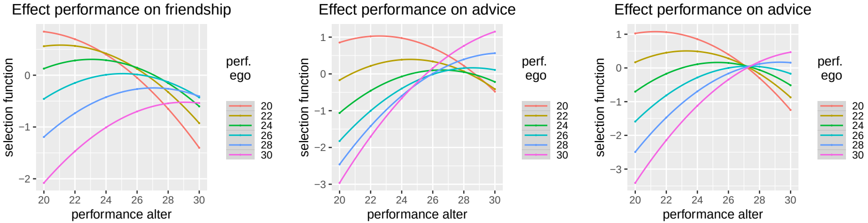
Effect	*Advice with Friendship* par.	(s.e.)	*Advice only* par.	(s.e.)
outdegree (density)	-3.751^{***}	(0.264)	-2.606^{***}	(0.212)
reciprocity	1.133^{***}	(0.245)	1.913^{***}	(0.228)
transitive triplets	0.210^{***}	(0.053)	0.309^{***}	(0.044)
transitive recipr. triplets	0.027	(0.090)	–0.022	(0.085)
indegree - popularity	0.0443^{***}	(0.0075)	0.036^{***}	(0.006)
outdegree - popularity	0.024	(0.027)	–0.050	(0.032)
outdegree - activity	0.050^{***}	(0.015)	0.017	(0.013)
reciprocated degree - activity	-0.118^{**}	(0.042)	-0.106^{**}	(0.041)
gender alter	0.027	(0.097)	0.010	(0.093)
gender ego	-0.202^{*}	(0.094)	-0.281^{**}	(0.094)
same gender	0.048	(0.091)	$0.163^{\dagger}$	(0.089)
same nationality	0.358^{**}	(0.121)	0.454^{***}	(0.119)
performance alter	0.139^{***}	(0.033)	0.080^{**}	(0.030)
performance ego	–0.014	(0.031)	-0.073^{*}	(0.030)
performance squared ego	0.043^{***}	(0.010)	0.029^{**}	(0.010)
performance difference squared	-0.0272^{***}	(0.0074)	-0.031^{***}	(0.007)

† $p < 0.1$; * $p < 0.05$; ** $p < 0.01$; *** $p < 0.001$;

Note:
for univariate advice results,
reciprocity, transitivity, and homophily parameters are higher;
in the more reasonable multivariate model,
these are ‘borrowed’ from the friendship network.

The results for performance are between those results
for friendship and advice in the co-evolution model.

Selection functions: comparison univariate and multivariate



Left, middle:

Co-evolution of friendship and advice: selection functions for performance; right: selection functions for performance on advice in univariate analysis.

Goodness of fit

For the friendship and advice networks separately,
the regular goodness of fit approach using `sienaGOF` can be used.

For the dependence between the two networks,
additional ‘auxiliary functions’ may be constructed.
One such function is available in RSiena: `mixedTriadCensus`,
implementing the mixed triad census
from Hollway, Lomi, Pallotti, and Stadtfeld (*Network Science*, 2017).

Motifs / mixed triads used in `mixedTriadCensus`.

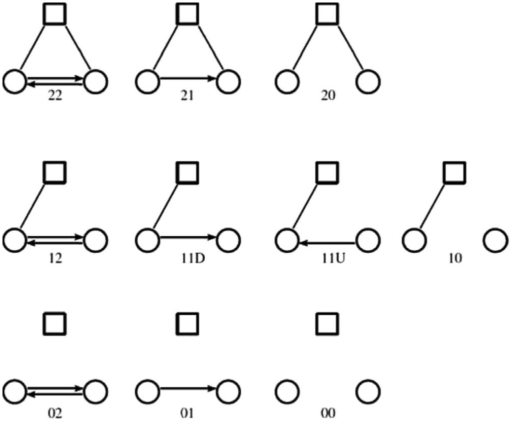


Figure 1 in Hollway, Lomi, Pallotti, and Stadtfeld (*Network Science*, 2017).

In this figure, ties between the bottom nodes are for the first network, ties from the bottom to the top nodes are for the second network, which can be two-mode. If the second network is one-mode, the set of triads considered is only a subset of all mixed triads, and ties in the figure are directed upward; existence of other ties is not considered.

The further analysis of this data set is not considered here.

Steps:

1. The fit of the mixed triad census was poor ($p = 0$).
2. If a theoretically valid model does not fit, time homogeneity should be tested first.
3. There was an important deviation from time homogeneity, mainly for friendship.
4. It turned out that there was an outlying actor (outdegrees 67, 71, 47) that was the main culprit (information in `summary` of `sienaTimeTest`, because an `egoX` effect was used for a dummy for this actor).
5. However, interacting the dummy variable for this actor with time led to divergence.
6. Therefore all row entries for this actor were defined as structural values.
7. The multivariate model for this data set did not have a good fit for the mixed triad census and also was not time-homogeneous.

Here I stopped for lack of time.

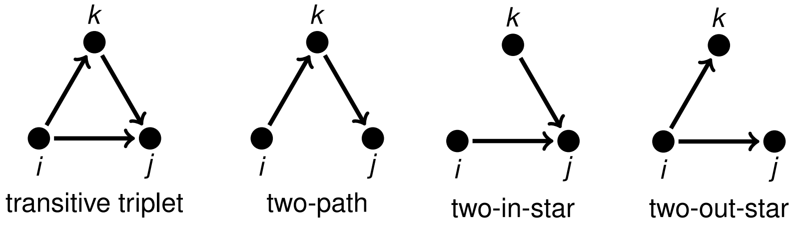
But this illustrates steps one may take if fit is not satisfactory.

Model specification: hierarchy requirements

For the SAOM there are hierarchy principles
somewhat like in regression analysis:
simpler configurations should be used as controls
for complicated configurations.

(Cf. regression analysis, where interaction effects are not interpretable
if the main effects are not included.)

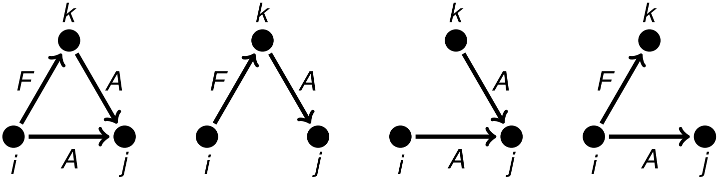
Hierarchy for univariate SAOMs



The transitive triplet (left) includes three subgraphs (right);
actor i can create a transitive triplet by closing $i \rightarrow j$ or $i \rightarrow k$;
therefore, to properly test transitivity, the two-path
and two-in-star configurations should be included in the model.
These correspond to the
outdegree-popularity and indegree-popularity effects.

In practice, this is even more serious for multiple network co-evolution.

To get evidence for mixed triadic effects, such as, e.g.,
'an advisor of a friend becomes an advisor',
you have to include (in this case) also two mixed degree effects
(mixed indegree activity and mixed outdegree activity for advice)
and indegree popularity for advice:



(F = Friendship; A = Advice)

This is to rule out alternative explanations: e.g.,
those who nominate a lot of friends might also nominate a lot of advisors.

7. Some final remarks

⇒ See Snijders, Lomi & Torlò in *Social Networks*, 2013.

⇒ Testing cross-network dependencies in dynamics of multiple networks gives interesting new possibilities for hypothesis testing.

7. Some final remarks

⇒ See Snijders, Lomi & Torlò in *Social Networks*, 2013.

⇒ Testing cross-network dependencies in dynamics of multiple networks gives interesting new possibilities for hypothesis testing.

⇒ Hierarchy should be respected in the model specification, if evidence for higher-order (e.g., triadic) phenomena is desired.

7. Some final remarks

- ⇒ See Snijders, Lomi & Torlò in *Social Networks*, 2013.
- ⇒ Testing cross-network dependencies in
  dynamics of multiple networks gives interesting
  new possibilities for hypothesis testing.
- ⇒ Hierarchy should be respected in the model specification,
  if evidence for higher-order (e.g., triadic) phenomena is desired.
- ⇒ Elaborated along the lines of actor-based modeling.

7. Some final remarks

⇒ See Snijders, Lomi & Torlò in *Social Networks*, 2013.

⇒ Testing cross-network dependencies in dynamics of multiple networks gives interesting new possibilities for hypothesis testing.

⇒ Hierarchy should be respected in the model specification, if evidence for higher-order (e.g., triadic) phenomena is desired.

⇒ Elaborated along the lines of actor-based modeling.

⇒ Compared to modeling dynamics of single networks, this approach attenuates the Markov assumption by extending the state space to a multiple network.

$\Rightarrow$ Further perspectives possible
by combining one-mode and two-mode networks.

⇒ Further perspectives possible
by combining one-mode and two-mode networks.

⇒ The method is available in RSiena.
This works for a small number (e.g., 2–6) of networks,
and a limited number of actors (up to a few hundred).

- ⇒ Further perspectives possible
  by combining one-mode and two-mode networks.
- ⇒ The method is available in RSiena.
  This works for a small number (e.g., 2–6) of networks,
  and a limited number of actors (up to a few hundred).
- ⇒ If there are implication relations between the networks,
  e.g., two networks might be mutually exclusive,
  or one might be a sub-network of the other,
  then this constraint is observed, noted in the `print01Report`, and
  respected in the simulations.
  This gives possibilities for networks with valued ties
  by using different dichotomies.