A.2 Poisson processes and birth processes

This sheet is for the week 4 class, not *the week 3 class. There will be one sheet per class.*

1. Let X be a Poisson process with rate $\lambda > 0$ and $X_0 = 0$. Denote $m(t) = \mathbb{E}(X_t)$.

   (a) Calculate $\mathbb{E}(X_t)$ by conditioning on the first arrival time $T_1 = \inf\{t \geq 0 : X_t = 1\}$. First show that

   $$m(t) = \int_0^t \lambda e^{-\lambda s} m(t-s) ds + 1 - e^{-\lambda t}$$

   Then make a change of variables $u = t - s$ and differentiate to deduce $m(t)$ explicitly. Is there a quicker way to calculate $m(t)$?

   (b) State the Markov property in terms of $(X_r)_{r \leq t}$ and $(X_{t+s})_{s \geq 0}$ and deduce that for $v < t < u$ and $k \in \mathbb{N}$, X_v is conditionally independent of X_u given $X_t = k$.

   (c) Calculate $\mathbb{P}(X_v = m | X_t = k)$ for $0 \leq m \leq k$ and $0 \leq v \leq t$. Determine the conditional distribution of X_v given $X_t = k$.

2. Each bacterium in a colony splits into two identical bacteria after an exponential time of parameter λ, which then split independently in the same way. Let X_t denote the size of the colony at time t, and suppose $X_0 = 1$.

   (a) Show that $(X_t)_{t \geq 0}$ has independent exponentially distributed holding times $(Z_n)_{n \geq 1}$. Deduce that it is a birth process and calculate the birth rates $(\lambda_n)_{n \geq 1}$.

   (b) Condition on the first splitting time to show that the probability generating function $\phi(t) = \mathbb{E}(z^{X_t})$, for any $z \in (-1, 1)$, satisfies

   $$\phi(t) = z e^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \phi(t-s)^2 ds$$

   Show that this implies $\phi'(t) = \lambda \phi(t)(\phi(t) - 1)$. Calculate the probability generating function of the geometric distribution with probability mass function $p_n = q^{n-1}(1-q)$, $n \geq 1$. Deduce that, for $q = 1 - e^{-\lambda t}$ and $n = 1, 2, \ldots$

   $$\mathbb{P}(X_t = n) = q^{n-1}(1-q).$$

   (c) Show that

   $$\mathbb{P}(X_u = k | X_0 = m) = \binom{k-1}{m-1} (1 - e^{-\lambda u})^{k-m} (e^{-\lambda u})^m$$

   for $u \geq 0$ and $1 \leq m \leq k$. Hint: Sums of independent geometric random variables have a so-called negative binomial distributions, which arise as the number of trials up to the nth head in a sequence of independent coin flips.

   (d) State a version of the Markov property for $(X_t)_{t \geq 0}$.

   (e) Use the Markov property at s to calculate $\mathbb{P}(X_s = m, X_t = k)$ and hence $\mathbb{P}(X_s = m | X_t = k)$ for $s \leq t$ and $1 \leq m \leq k$.

3. Let $(X_t)_{t\geq 0}$ be a simple birth process starting from $X_0 = i \geq 0$.
   (a) Formulate and prove the strong Markov property at the stopping time $T_{\{j\}} = \inf\{t \geq 0 : X_t = j\}$, i.e. the first hitting time of $j \in \mathbb{N}$ for some $j > i$.
   (b) In a libertine population any pair of individuals have single offspring at rate λ (do not distinguish male and female; any individual will form a pair with each of the other individuals!). Let $X_0 = 2$. Denote the population size at time t by X_t. Show that $(X_t)_{t\geq 0}$ is a birth process. Calculate the birth rates $(\lambda_n)_{n\geq 2}$ and show that X is explosive. Calculate $\mathbb{E}(T_\infty)$. Hint: $1/k - 1/(k+1) = 1/(k(k+1))$.

M.Sc. students and keen undergraduates should also try to solve the following exercises.

4. Assume a population model where each individual gives birth independently and repeatedly at rate λ. Assume furthermore, that more individuals immigrate according to a Poisson process with rate ν. Suppose $X_0 = 0$.
   (a) Specify rates $(\lambda_n)_{n\geq 0}$ of a birth process to model this.
   (b) Show that $m(t) = \mathbb{E}(X_t)$ satisfies
   $$m(t) = \int_0^t m(t-s)\nu e^{-\nu s}ds + \frac{\nu}{\lambda + \nu}\left(e^{\lambda t} - e^{-\nu t}\right).$$
   (c) Solve this integral equation to find $m(t)$.

5. Suppose you have a large number of m atoms from a radio-active substance, and you have a Geiger counter that can detect all particles emitted (one from each atom). Each atom emits its particle at an $Exp(\lambda)$ time independent of the others. Denote by X_t the number of particles detected before time t.
   (a) Argue that X is a birth process and determine λ_n, $0 \leq n < m$.
   (b) Comment on the suggestion to put $\lambda_n = 0$ for $n \geq m$. How can this be interpreted? What is an exponential random variable with parameter 0?
   (c) Determine whether X is explosive. How does this fit with the explosion criterion for birth processes (Proposition 16)?

6. (a) Let X be a process in which births occur at rates $(\lambda_n)_{n\geq 0}$, but each birth is only successful with probability $p \in (0,1)$, independently for each birth. Otherwise the population size does not change. Show that X is a birth process with rates $(p\lambda_n)_{n\geq 0}$. Hint: fix a population size n and consider the (random) number of births until a birth is successful.
   (b) One problem with real Geiger counters is that they only detect a fraction $p \in (0,1)$ of particles. In the setting of Exercise 5, is the modified particle counting process still a birth process? Hint: consider the special case $m = 1$ and calculate the distribution of the first count.
   Relate this to (a).