

Restricted exchangeable partitions and embedding of associated hierarchies in continuum random trees

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Abstract

We introduce the notion of a restricted exchangeable partition of \mathbb{N} and study natural classes of such partitions. We obtain integral representations, study associated coalescents and fragmentations, embeddings into continuum random trees and convergence to such limit trees. As an application, we deduce from the general theory developed here a particular limit result conjectured previously for Ford's alpha model and its non-binary extension, the alpha-gamma model, where restricted exchangeability arises naturally.

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1 Introduction

Following de Finetti and Kingman, we call a measure on the space \mathcal{P}_B of partitions of $B \subseteq \mathbb{N}$ *exchangeable*, if it is invariant under the natural action on \mathcal{P}_B of the symmetric group on B ; and a random partition is called exchangeable if its distribution is exchangeable. For a partition $\pi = \{\pi_i, i \in \mathbb{N}\}$, each non-empty $\pi_i \subseteq B$ is called a *block* of π . When π has only finitely many blocks, we often omit \emptyset from π . We arrange the blocks of π in the order of least element, i.e. $\min \pi_i < \min \pi_j$ for every $i < j$, followed by \emptyset with the convention $\min \emptyset = \infty$ to be definite. For finite π_i , we consider the *block size* $\#\pi_i$. For $n \in \mathbb{N}$, we set $[n] = \{1, \dots, n\}$. Then a measure μ on $\mathcal{P} = \mathcal{P}_{\mathbb{N}}$ is exchangeable if and only if the discrete measures μ_n on $\mathcal{P}_n = \mathcal{P}_{[n]}$, given by

$$\mu_n(\{\pi\}) = \mu(\mathcal{P}^\pi), \quad \pi \in \mathcal{P}_n, \text{ where } \mathcal{P}^\pi = \{\Gamma \in \mathcal{P} : \Gamma|_n = \pi\} \text{ and } \Gamma|_n = \{\Gamma_i \cap [n], i \geq 1\}, \quad (1)$$

are exchangeable for all $n \geq 1$. Furthermore, a measure μ_n on \mathcal{P}_n is exchangeable if $\mu_n(\{\pi\}) = \mu_n(\{\pi'\})$ for all $\pi, \pi' \in \mathcal{P}_n$ with the same multiset of block sizes.

Several weaker forms of exchangeability have been studied in the literature, notably Pitman's partial exchangeability [28] and Gnedin's constrained exchangeability [15]. See Section 3.1. We introduce here a new weak form of exchangeability. We first call a measure μ on a subset $\mathcal{S}_B \subseteq \mathcal{P}_B$ of partitions of a finite $B \subset \mathbb{N}$ *exchangeable on \mathcal{S}_B* if $\mu(\{\pi\}) = \mu(\{\pi'\})$ for all $\pi, \pi' \in \mathcal{S}_B$ with the same multiset of block sizes. Now consider the infinite case.

Definition 1 Let $\mathcal{S} \subseteq \mathcal{P}$. Consider $\mathcal{S}_n = \{\pi \in \mathcal{P}_n : \mathcal{P}^\pi \subseteq \mathcal{S}\}$, $n \geq 1$. We call a measure μ on \mathcal{S} *exchangeable on \mathcal{S}* if the restrictions to \mathcal{S}_n of the measure μ_n given by (1) are exchangeable on \mathcal{S}_n , $n \geq 1$, and if $\{\Gamma \in \mathcal{S} : \Gamma|_n \notin \mathcal{S}_n \text{ for all } n \geq 1\}$ is a μ -null set.

A measure μ on \mathcal{P} is called *restricted exchangeable* if \mathcal{P} can be decomposed into disjoint measurable \mathcal{P}^j , $j \geq 0$, so that the restrictions of μ to \mathcal{P}^j are finite and exchangeable on \mathcal{P}^j .

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Remark 2 (i) Alternatively, we might include in \mathcal{S}_n all partitions π of \mathcal{P}_n whose associated cylinder sets \mathcal{P}^π intersect \mathcal{S} . When this makes a difference, some of the \mathcal{P}_n^j , $j \geq 0$, in the definition of restricted exchangeability would not be disjoint, giving a different notion that seems less natural to us and that we do not propose to investigate further in this paper.

(ii) If we dropped the requirement that the parts of \mathcal{S} missed by all μ_n , $n \geq 1$, form a μ -null set, then all (finite) measures on \mathcal{P} would be restricted exchangeable; we could choose $\mathcal{P}^j = \{\Gamma_j\}$, $j \geq 1$, $\mathcal{P}^0 = \mathcal{P} \setminus \{\Gamma_j, j \geq 1\}$ for a countable subset $\{\Gamma_j, j \geq 1\} \subset \mathcal{P}$, that is dense for the metric $d(\Gamma, \Gamma') = 2^{-\inf\{n \geq 1: \Gamma|_n \neq \Gamma'|_n\}}$, because then $\mathcal{P}_n^j = \emptyset$ for all $n \geq 1$.

It will be useful to consider Kingman's branching graph (\mathcal{K}, E) with $\mathcal{K} = \bigcup_{n \geq 0} \mathcal{P}_n$ rooted at the unique element $\{\emptyset\} \in \mathcal{P}_0$, equipped with the directed edge relation $(\pi', \pi) \in E$ if $\pi' \in \mathcal{P}_n$, $\pi \in \mathcal{P}_{n+1}$ with $\pi' = \pi \cap [n]$ for some $n \geq 0$, cf. [23]. Then $b: \mathcal{P} \rightarrow \mathcal{K}^{\mathbb{N}}$ given by $b(\Gamma) = (\Gamma|_n, n \geq 0)$ is an injection onto the set $\mathcal{G} \subset \mathcal{K}^{\mathbb{N}}$ of infinite chains starting at $\{\emptyset\}$. We write $\pi' \preceq \pi$ if $\pi' \in \mathcal{P}_n$ and $\pi' = \pi \cap [n]$. For $\pi \in \mathcal{K}$, we denote by $\mathcal{K}^\pi = \{\pi' \in \mathcal{K} : \pi' \preceq \pi\}$ the cylinder set of π in \mathcal{K} . We consider *subgraphs* $\mathcal{E} \subset \mathcal{K}$ of (\mathcal{K}, E) that, without further mentioning, are connected and contain $\{\emptyset\}$. For a subgraph \mathcal{E} all connected components of $\mathcal{K} \setminus \mathcal{E}$ are of the form \mathcal{K}^π for some $\pi \in \mathcal{K}$, indeed they are $\{\mathcal{K}^\pi : \pi \in \mathcal{C}_\mathcal{E}\}$, where $\mathcal{C}_\mathcal{E} = \{\pi \in \mathcal{K} \setminus \mathcal{E} : (\pi', \pi) \in E \text{ for some } \pi' \in \mathcal{E}\}$.

Let $S^\downarrow = \{\mathbf{s} = (s_i, i \geq 1) : s_1 \geq s_2 \geq \dots \geq 0, \sum_{i \geq 1} s_i \leq 1\}$. For $\mathbf{s} \in S^\downarrow$, Kingman's paintbox [24] is the exchangeable distribution $\kappa_\mathbf{s}$ on \mathcal{P} of the value partition induced by independent random variables $(\xi_r, r \geq 1)$ with respective distributions

$$\mathbb{P}(\xi_r = i) = s_i, \quad i \geq 1, \quad \mathbb{P}(\xi_r = -r) = s_0 := 1 - \sum_{i \geq 1} s_i.$$

For $\pi \in \mathcal{K}$ we introduce here modified paintboxes by conditioning on the cylinder set $\mathcal{P}^\pi = \{\Gamma \in \mathcal{P} : \Gamma|_n = \pi\}$ of π in \mathcal{P} , but note that this conditioning is degenerate in some cases; specifically, for $\mathbf{s} \in S^\downarrow$ let $m \geq 0$ such that $s_m > s_{m+1} = 0$ (or $m = \infty$ if $s_i > 0$ for all $i \geq 1$), suppose that $\pi \in \mathcal{K}$ has k non-empty blocks, of which ℓ are of size at least 2, then we call *admissible for* (π, π', \mathbf{s}) , $\pi' \succeq \pi$, any collection $(i_1, \dots, i_{k'})$ of indices that are

- distinct with $i_j \geq 1$ except that we allow $i_j = 0$ for any (also multiple) j with $\#\pi'_j = 1$; this applies if $s_0 > 0$ and $m \geq \ell$, or $s_0 = 0$ and $m \geq k$, (non-degenerate case);
- distinct with $i_j \geq 1$ except that we allow $i_j = 0$ for any j with $\pi'_j = 1$ or $\#\pi'_j = \#\pi_j < r$ and also for $q - m$ of the j with $\#\pi'_j = \pi_j = r$, where (q, r) is such that $< m$ blocks of π have size $\geq r + 1$ and $q \geq m$ have size $\geq r$; this applies otherwise (degenerate case).

The modified paintboxes are now given as measures on $\mathcal{P}^\pi \subset \mathcal{P}$, for $\pi' \in \mathcal{K}^\pi$, by

$$\kappa_\mathbf{s}^\pi \left(\mathcal{P}^{\pi'} \right) = \frac{1}{Z_\mathbf{s}^\pi} \sum_{(i_1, \dots, i_{k'}) \text{ admissible for } (\pi, \pi', \mathbf{s})} s_0^{\#\{1 \leq j \leq k' : i_j = 0\} - (k-m)^+} \prod_{1 \leq j \leq k' : i_j \neq 0} s_{i_j}^{\#\pi'_j},$$

where $Z_\mathbf{s}^\pi$ is the normalisation constant; we can drop $(k-m)^+ = (k-m)1_{\{k > m\}}$ unless $s_0 = 0$. For $\pi = \{\{1\}\}$, this is a well-known formula for Kingman's paintbox $\kappa_\mathbf{s} = \kappa_\mathbf{s}^\pi$, with $Z_\mathbf{s}^\pi = 1$.

Theorem 3 *Let μ be a measure on \mathcal{P} . Then μ is restricted exchangeable if and only if there are a subgraph $\mathcal{E} \subset \mathcal{K}$ and for each $\pi \in \mathcal{C}_\mathcal{E}$ a finite measure ν_π on S^\downarrow such that*

$$\mu = \sum_{\pi \in \mathcal{C}_\mathcal{E}} \int_{S^\downarrow} \kappa_\mathbf{s}^\pi \nu_\pi(d\mathbf{s}).$$

Note that restricted exchangeable measures μ can be infinite, if \mathcal{E} is infinite. In this case, we can consider the pre-image \mathcal{P}^0 of the set of the infinite chains in \mathcal{E} under the bijection $b: \mathcal{P} \rightarrow \mathcal{G}$ introduced above. Then $\mu(\mathcal{P}^0) = 0$ and μ is finite on all compact subsets of $\mathcal{P} \setminus \mathcal{P}^0$, in the sense induced by the metric topology on \mathcal{P} described in Remark 2(ii), in particular μ is then σ -finite. Indeed, such measures are locally finite in the sense of Vershik and Kerov [34].

Examples 4 (i) A natural class of restricted exchangeable measures can be obtained by conditioning an exchangeable random partition Π on $\{\Pi \cap [n] = \pi\}$ for some $\pi \in \mathcal{P}_n$. More generally, for $\mathcal{E} = \bigcup_{k=0}^n \mathcal{P}_k$, we can use total masses of ν_π to specify $\mathbb{P}(\Pi \cap [n] = \pi) = \nu_\pi(S^\downarrow)$ and given $\{\Pi \cap [n] = \pi\}$, specify asymptotic frequencies according to $\nu_\pi/\nu_\pi(S^\downarrow)$, $\pi \in \mathcal{P}_n$. We can use this idea to approximate any distribution μ on \mathcal{P} by restricted exchangeable distributions using $\nu_\pi(S^\downarrow) = \mu(\mathcal{P}^\pi)$, $\pi \in \mathcal{P}_n$, and any asymptotic frequencies, $n \geq 1$.

(ii) For $B \subseteq \mathbb{N}$, let $\mathbf{1}_B$ be the trivial partition of a single block B . *Dislocation measures* are measures on $\mathcal{P} \setminus \{\mathbf{1}_\mathbb{N}\}$, finite on compact subsets. See Section 3.2. We set $\mathcal{E} = \{\mathbf{1}_{[j]}, j \geq 1\}$ and note that the connected components of $\mathcal{K} \setminus \mathcal{E}$ are $\mathcal{K}^j = \mathcal{K}^{\{[j], \{j+1\}\}}$, $j \geq 1$, so that $\mathcal{P}^j = \mathcal{P}^{\{[j], \{j+1\}\}}$, $j \geq 1$, is a natural decomposition of $\mathcal{P} \setminus \{\mathbf{1}_\mathbb{N}\}$. Bertoin's [6] (possibly infinite) exchangeable dislocation measures are clearly exchangeable on \mathcal{P}^j , $j \geq 1$; also the finiteness condition on \mathcal{P}^j holds since \mathcal{P}^j is compact for all $j \geq 1$.

(iii) Ford's alpha model [14] and the alpha-gamma Markov branching model [11] are restricted exchangeable, but not exchangeable. See Section 6 for the definition of these natural examples of restricted exchangeable dislocation measures as well as conclusions.

(iv) For $B \subseteq \mathbb{N}$, let $\mathbf{0}_B$ be the partition of B into singleton blocks $\{j\}$, $j \in B$. Coalescent L -measures (allowing simultaneous multiple collisions) are measures on $\mathcal{P} \setminus \{\mathbf{0}_\mathbb{N}\}$, finite on compact subsets. See Section 3.2. Set $\mathcal{E} = \{\mathbf{0}_{[n]}, n \geq 1\}$, $\mathcal{K}^{(k-1)(k-2)/2+i} = \mathcal{K}^{\pi_{ik}}$, where $\pi_{ik} = (\mathbf{0}_{[k-1]} \setminus \{\{i\}\}) \cup \{\{i, k\}\}$, and $\mathcal{P}^{(k-1)(k-2)/2+i} = \mathcal{P}^{\pi_{ik}}$ for $1 \leq i < k$. Schweinsberg's [32] (possibly infinite) exchangeable L -measures are finite and exchangeable on \mathcal{P}^j , $j \geq 1$.

From Theorem 3 we deduce an integral representation for restricted exchangeable dislocation measures. For simplicity we only allow as decomposition of \mathcal{P} in Definition 1 the most relevant and natural $\mathcal{P}^0 = \{\mathbf{1}_\mathbb{N}\}$ and $\mathcal{P}^j = \mathcal{P}^{\{[j], \{j+1\}\}}$, $j \geq 1$.

Corollary 5 *Let κ be a restricted exchangeable measure with subgraph $\mathcal{E} = \{\mathbf{1}_{[j]}, j \geq 1\}$. Then for each $j \geq 1$, there are constants $c_j \geq 0$ and $k_j \geq 0$, and a measure ν_j on S^\downarrow with*

$$\nu_j(\{(0, 0, \dots)\}) = \nu_j(\{(1, 0, \dots)\}) = 0 \quad \text{and} \quad \int_{S^\downarrow} \left(s_0 \mathbf{1}_{\{j=1\}} + \sum_{i \geq 1} s_i^j (1 - s_i) \right) \nu_j(ds) < \infty,$$

such that, for $\varepsilon^{(j)} = \{\{j\}, \mathbb{N} \setminus \{j\}\}$ and $\omega^{[j]} = \{[j], \{j+1\}, \{j+2\}, \dots\}$, $j \geq 1$,

$$\kappa = c_1 \delta_{\varepsilon(1)} + \sum_{j \geq 1} \left(c_j \delta_{\varepsilon(j+1)} + k_j \delta_{\omega^{[j]}} + \int_{S^\downarrow} \kappa_s(\cdot \cap \mathcal{P}^j) \nu_j(ds) \right), \quad \text{where } \mathcal{P}^j = \mathcal{P}^{\{[j], \{j+1\}\}}.$$

Bertoin [6] iterated random partitions from an exchangeable dislocation measure ν to create exchangeable \mathcal{P} -valued fragmentation processes $(F^*(t), t \geq 0)$. Furthermore, the associated closed exchangeable hierarchies $\mathcal{H}^* = \{F_i^*(t), i \geq 1, t \geq 0\}^{\text{cl}}$ of blocks visited by such a process are naturally interpreted as trees, see e.g. [26], and are often naturally embedded in α -self-similar continuum random trees (CRTs) [19], say $(\mathcal{T}_{(\alpha, \nu)}, \mu)$, via samples $\Sigma_i^* \in \mathcal{T}_{(\alpha, \nu)}$, $i \geq 1$, conditionally independent given $(\mathcal{T}_{(\alpha, \nu)}, \mu)$, each with distribution μ , as $\mathcal{H}^* = \{\mathcal{L}^*(\mathcal{T}^v), v \in \mathcal{T}_{(\alpha, \nu)}\}^{\text{cl}}$, where $\mathcal{L}^*(\mathcal{T}^v) = \{i \in \mathbb{N} : \Sigma_i^* \in \mathcal{T}^v\}$ and \mathcal{T}^v is the subtree of $\mathcal{T}_{(\alpha, \nu)}$ rooted at v . See Sections 3.2 and 4.1. Vice versa, such exchangeable hierarchies derived from fragmentation processes (or Markov branching trees) can be used to construct CRTs as scaling limits [20]. In the second part of this paper we carry out a similar programme for the restricted exchangeable case, starting from a restricted exchangeable dislocation measure of the form identified in Corollary 5.

One of our main results is a general embedding result. This generalises [30, Theorem 4], which treated Ford's alpha model and a binary two-parameter extension that, apart from the alpha model, is not restricted exchangeable in the sense of Corollary 5.

Theorem 6 *Let \mathcal{H} be the hierarchy of a restricted exchangeable fragmentation such that*

$$\nu(ds) = \sum_{j \geq 1} \left(\sum_{i \geq 1} s_i^j (1 - s_i) \right) \nu_j(ds)$$

satisfies $\int_{S^1} (1 - s_1) \nu(ds) < \infty$ and $\nu(s_0 > 0) = 0$, and such that $c_j = k_j = 0$ for all $j \geq 1$. Then \mathcal{H} can be embedded as $\mathcal{H} = \{\mathcal{L}(\mathcal{T}^\nu), \nu \in \mathcal{T}_{(\alpha, \nu)}\}^{\text{cl}}$ in an α -self-similar CRT $\mathcal{T}_{(\alpha, \nu)}$ with dislocation measure ν , where $\mathcal{L}(\mathcal{T}^\nu) = \{i \in \mathbb{N} : \Sigma_i \in \mathcal{T}^\nu\}$ for some (dependent) $\Sigma_i \in \mathcal{T}_{(\alpha, \nu)}$, $i \geq 1$.

There is an integral representation for coalescent L -measures analogous to Corollary 5, see Propositions 13 in Section 3.3. We leave open the question whether associated restricted exchangeable coalescents can be embedded in the Λ -coalescent measure trees (and analogous trees for Ξ -coalescents) in the sense of Greven et al. [17].

Let us turn to the convergence to CRTs. Just like partitions, hierarchies of \mathbb{N} are uniquely determined by their restrictions $T_n = \mathcal{H}|_n = \{B \cap [n], B \in \mathcal{H}\}$ that form here a consistent family $(T_n, n \geq 1)$ of random trees with T_n as vertex set and implicit edge set given by the parent-child relation that assigns to each non-singleton $B \in T_n$ as children the maximal strict subsets in T_n that, by construction, form a partition of B . We can delabel these trees and add a root vertex to obtain rooted combinatorial trees T_n° , i.e. connected acyclic graphs with no degree-2 vertex, but some degree-1 vertices, one of which is the distinguished root. We can regard T_n° as a path-connected metric space with unit distance between vertices and connected by unit line segments. Notation T_n°/a is then understood as scaling all distances by a factor $a \in (0, \infty)$.

In the exchangeable case, [20] obtain CRT convergence under a regular variation condition

$$\nu(s_1 \leq 1 - \varepsilon) = \varepsilon^{-\alpha} \ell(1/\varepsilon) \quad \text{as } \varepsilon \downarrow 0; \text{ for some } \alpha \in (0, 1) \text{ and slowly varying } \ell \quad (2)$$

and a log-moment condition

$$\int_{S^1} \sum_{i \geq 2} s_i |\log(s_i)|^\varrho \nu(ds) < \infty \quad \text{for some } \varrho > 0. \quad (3)$$

Theorem 7 *If in the setting of Theorem 6, the dislocation measure ν satisfies (2) and (3), and if $\nu_j = \nu_m$ for some $m \geq 1$ and all $j \geq m$, then*

$$\frac{T_n^\circ}{n^\alpha \ell(n) \Gamma(1 - \alpha)} \rightarrow \mathcal{T}_{(\alpha, \nu)} \quad \text{in probability, in the Gromov-Hausdorff sense.}$$

In addition to the proofs of the main results formulated in the Introduction, the content of this paper is as follows. Section 2 mainly proves Theorem 3 and Corollary 5. Section 3 includes a discussion of the relationship between restricted exchangeability, partial exchangeability and constrained exchangeability, a more detailed introduction to restricted exchangeable coalescents and fragmentations and associated hierarchies, and an integral representation analogous to Corollary 5 for restricted exchangeable coalescents.

Sections 4 and 5 deal with the proofs of Theorems 6 and 7, but we also develop a general method to sample leaves in non-binary self-similar CRTs in Section 4, while Section 5 studies in some detail the embedding used to prove Theorem 6 in order to obtain individual estimates for each leaf Σ_i , $i \geq 1$, where in the proof in [20] for the exchangeable case, consideration of a single leaf Σ^* in $(\mathcal{T}_{(\alpha, \nu)}, \mu)$ sampled from μ gives direct estimates for all Σ_i^* , $i \geq 1$. This includes here estimates for a p th moment renewal theorem in Lemma 23 and an application to Gnedin's constrained paintboxes in Lemma 24, both of which may be of independent interest. While we build on [20], we do not repeat the content of [20], we rather focus on the new developments here

and then provide only a half-page sketch when the arguments of [20] can be applied to complete the proof of Theorem 7, with the exception of Lemma 27, which deals with large deviations of block numbers of partitions associated with subordinators and may be of independent interest. Its proof is completely analogous but more general than in [20] and references therein, so we have rewritten it for the present context and included it in an appendix. Significant intermediate results for the proof of Theorem 7 include almost sure convergences of rescaled subtrees of T_n spanned by k leaves as first $n \rightarrow \infty$ in Proposition 28 and then also $k \rightarrow \infty$ in Formula (23).

Section 6 demonstrates how the main results obtained in this paper can be applied to the alpha model, the alpha-gamma model and more general skewed Poisson-Dirichlet models that we introduce here as a natural three-parameter family of restricted exchangeable fragmentation models. We show in these examples that Theorems 6 and 7 typically refer to Markov branching trees T_n° that are not sampling consistent, so that the theory developed in [20] does not even yield convergence in distribution for these trees, where we here establish convergence in probability.

2 Integral representations, proof of Theorem 3 and Corollary 5

Our first aim is to understand exchangeability on subsets of the form $\mathcal{P}^\pi \subseteq \mathcal{P}$ for some $\pi \in \mathcal{P}_n$. It is easy to show that the modified paintboxes κ_s^π are exchangeable on \mathcal{P}^π .

Proposition 8 *The modified paintbox κ_s^π can be expressed in terms of any $\Gamma \in \mathcal{P}^\pi$ with asymptotic frequencies \mathbf{s} , provided that any blocks of Γ with zero asymptotic frequency are either subsets of $[n]$ or singletons, as*

$$\kappa_s^\pi(\mathcal{P}^{\pi'}) = \lim_{r \rightarrow \infty} \frac{\#\{\pi'' \in \mathcal{P}_r^{\pi'} : \pi'' \sim \Gamma|_r\}}{\#\{\pi'' \in \mathcal{P}_r^\pi : \pi'' \sim \Gamma|_r\}},$$

where $\pi' \sim \pi''$ if π' and π'' have the same multiset of block sizes.

Proof. This proof is a refinement of the relevant part of the proof of [23, Theorem 3.1], Kerov's proof of Kingman's paintbox representation of exchangeable partitions in \mathcal{P} , where we need to take into account the distortions due to restriction to \mathcal{P}^π . We evaluate the right-hand side. Numerator and denominator are easily calculated as

$$\#\{\pi'' \in \mathcal{P}_r^{\pi'} : \pi'' \sim \Gamma|_r\} = \sum \binom{r-n'}{\#\Gamma_{i_1}|_r - \#\pi'_1, \dots, \#\Gamma_{i_{k'}}|_r - \#\pi'_{k'}, \#\Gamma_{\text{others}}|_r} \frac{1}{\prod_{j=1}^{\infty} p_j!},$$

where \sum is over indices $(i_1, \dots, i_{k'})$ such that $\#\Gamma_{i_j}|_r - \#\pi'_j \geq 0$ for all $j \in [k']$, $\Gamma_{\text{others}}|_r$ is the vector of all $\Gamma_i|_r$, $i \geq 1$, except $\Gamma_{i_1}|_r, \dots, \Gamma_{i_{k'}}|_r$ and p_j is the number of blocks of $\Gamma|_r$ with j elements, $j \geq 1$. Now first assume $s_0 = 1 - \sum_{i \geq 1} s_i = 0$, then we deduce that the limit exists and is given by $Z_s^{\pi, \pi'} / Z_s^{\pi, \pi}$, where

$$Z_s^{\pi, \pi'} = \lim_{r \rightarrow \infty} \frac{\sum \binom{r-n}{\#\Gamma_{i_1}|_r - \#\pi'_1, \dots, \#\Gamma_{i_{k'}}|_r - \#\pi'_{k'}, \#\Gamma_{\text{others}}|_r} r^d}{\binom{r}{\#\Gamma_1|_r, \#\Gamma_2|_r, \dots}} = \sum_{(i_1, \dots, i_{k'}) \text{ admissible for } (\pi, \pi', s)} \prod_{j: s_{i_j} > 0} s_{i_j}^{\#\pi'_j},$$

where d is the minimal sum over all those block sizes that have to be mapped to zero-limiting-frequency blocks (if any; $d > 0$ only in the degenerate case); in fact this power d is such that terms with higher than the minimal sum actually become negligible, and we identify $\kappa_s^\pi(\mathcal{P}^{\pi'})$.

If $s_0 > 0$, blocks of zero limiting frequency need to be treated differently, because their union $\tilde{\Gamma}_0$ now has a limiting frequency, and a union $\tilde{\pi}'_0$ of blocks of π' can indeed be associated with $\tilde{\Gamma}_0$. Specifically, we calculate a first factor as

$$\lim_{r \rightarrow \infty} \frac{\sum \binom{r-n}{\#\tilde{\Gamma}_0 - \#\tilde{\pi}'_0, \#\tilde{\Gamma}_{i_1}|_r - \#\tilde{\pi}'_1, \dots, \#\tilde{\Gamma}_{i_{\tilde{k}'}}|_r - \#\tilde{\pi}'_{\tilde{k}'}, \#\tilde{\Gamma}_{\text{others}}|_r} r^d}{\binom{r}{\#\tilde{\Gamma}_0|_r, \#\tilde{\Gamma}_1|_r, \#\tilde{\Gamma}_2|_r, \dots}} = \sum_{(i_1, \dots, i_{\tilde{k}'}) \text{ admissible for } (\tilde{\pi}, \tilde{\pi}', s)} s_0^{\#\tilde{\pi}'_0} \prod_{j=1}^{\tilde{k}'} s_{i_j}^{\#\tilde{\pi}'_j},$$

but then need to also count the further partitions of the block of size $\#\tilde{\Gamma}_0$. This yields for $\tilde{\pi}'_0 = \pi'_{j_1} \cup \dots \cup \pi'_{j_b}$ a positive limit factor if $d = \#\tilde{\pi}'_0 - b$ is minimal, which we then calculate as

$$\lim_{r \rightarrow \infty} \frac{\sum \binom{\#\tilde{\Gamma}_0|_r - \#\tilde{\pi}'_0}{\#\Gamma_{i_1}|_r - \#\pi'_{j_1}, \dots, \#\Gamma_{i_b}|_r - \#\pi'_{j_b}, 1, \dots, 1} r^d}{(\#\tilde{\Gamma}_0|_r)!} = s_0^{-d};$$

note that the number of available indices is asymptotically equivalent to $\#\tilde{\Gamma}_0|_r \sim s_0 r$, so that the sum contains $\sim (\#\tilde{\Gamma}_0|_r)^b$ terms, and this contributes significantly to the asymptotics of the numerator. Finally we sum over the different choices of $\tilde{\pi}'_0$ with $\#\tilde{\pi}'_0 - b = d$ to identify $\kappa_s^\pi(\mathcal{P}^{\pi'})$. \square

With these representations of the modified paintboxes, we now obtain the integral representation of general measures that are exchangeable on \mathcal{P}^π for some $\pi \in \mathcal{P}_n$.

Proposition 9 *Let μ be a finite measure, exchangeable on \mathcal{P}^π for some $\pi \in \mathcal{P}_n$. Then there is a finite measure ν on S^\downarrow such that $\mu = \int_{S^\downarrow} \kappa_s^\pi \nu(ds)$.*

Proof. This proof uses a combination of the martingale method due to Vershik and Kerov [34, Theorem 2] and the de Finetti method used by Aldous [1]. W.l.o.g., μ is a probability measure. Let $\Pi \sim \mu$ for an exchangeable probability measure on \mathcal{P}^π . Consider the process

$$X_r = \frac{\#\{\pi'' \in \mathcal{P}_r^{\pi'} : \pi'' \sim \Pi|_r\}}{\#\{\pi'' \in \mathcal{P}_r^\pi : \pi'' \sim \Pi|_r\}}, \quad r \geq n',$$

in the decreasing filtration \mathcal{F}_r generated by the block sizes of $\Pi|_{r'}$, $r' \geq r$. By exchangeability, X_r only depends on its block sizes B_r and is hence \mathcal{F}_r -measurable and $\mathbb{E}[X_r | \mathcal{F}_{r+1}]$ only depends on X_{r+1} . For a multiset b of block sizes, denote by $m(b)$ (resp. $m'(b)$) the number of chains in \mathcal{K}^π from π (resp. in $\mathcal{K}^{\pi'}$ from π') to block sizes b . By exchangeability, each of these chains is equally likely. For block sizes $B_{r+1} = b_{r+1}$, we denote by $m(b_r, b_{r+1})$ the number of chains from a specific partition with block sizes b_r to any partition with block sizes b_{r+1} , then $m(b_r)m(b_r, b_{r+1})$ chains from π to block sizes b_{r+1} pass via block sizes b_r . With this notation, we have $X_r = m'(B_r)/m(B_r)$. Then

$$\mathbb{E}[X_r | B_{r+1} = b_{r+1}] = \sum_{b_r} \frac{m(b_r)m(b_r, b_{r+1})}{m(b_{r+1})} \frac{m'(b_r)}{m(b_r)} = \frac{1}{m(b_{r+1})} \sum_{b_r} m(b_r, b_{r+1})m'(b_r) = \frac{m'(b_{r+1})}{m(b_{r+1})}$$

for all admissible b_{r+1} shows that $(X_r, r \geq n)$ is a bounded martingale and hence converges a.s.

On the other hand, de Finetti's theorem yields that asymptotic frequencies exist μ -a.s. Specifically, consider a partition Π with distribution μ and, independently a sequence U_i , $i \geq 1$, of auxiliary independent uniform random variables. Then the random variables

$$X_j = U_i \quad \text{if } j \in \Pi_i, \quad j \geq n+1,$$

are exchangeable. By de Finetti's theorem, they are conditionally i.i.d. and the atom sizes S_i of the random limiting distribution in size-biased order satisfy

$$S_i = \lim_{r \rightarrow \infty} \frac{\#\{j \in \{n+1, \dots, n+r\} : X_j = U_i\}}{r} = \lim_{r \rightarrow \infty} \frac{\#\Pi_i \cap [r]}{r}.$$

Clearly, the latter limit does not depend on the auxiliary variables $(U_i, i \geq 1)$, so asymptotic frequencies exist μ -a.s. Furthermore, μ -a.e. partition is such that blocks with zero asymptotic frequency either only involve elements of $[n]$ or are singletons. Denote by ν the distribution on S^\downarrow of the asymptotic frequencies $(S_i, i \geq 1)$ rearranged into decreasing order of Π .

This means that μ is concentrated on those partitions for which Proposition 8 yields modified paintbox representations, and we see that $X_r \rightarrow \kappa_{\mathbf{S}}^{\pi}(\mathcal{P}^{\pi'})$ a.s., where $\mathbf{S} \sim \nu$; but $(X_r, r \geq n')$ is a bounded martingale, so exchangeability on \mathcal{P}^{π} yields

$$\int_{S^{\downarrow}} \kappa_{\mathbf{S}}^{\pi}(\mathcal{P}^{\pi'}) \nu(ds) = \mathbb{E}[\kappa_{\mathbf{S}}^{\pi}(\mathcal{P}^{\pi'})] = \mathbb{E}[X_{n'}] = \sum_{\tilde{\pi} \in \mathcal{P}_{n'}^{\pi'} : \tilde{\pi} \sim \pi'} \mu(\mathcal{P}^{\tilde{\pi}}) \frac{1}{\#\{\pi'' \in \mathcal{P}_{n'}^{\pi'} : \pi'' \sim \tilde{\pi}\}} = \mu(\mathcal{P}^{\pi'}). \quad \square$$

This proof raises the question whether we could have done without the martingale method or without the de Finetti argument, as can be done in the exchangeable case. In fact, to avoid de Finetti, we would have to generalise Proposition 8 to ensure that all Γ for which the limits in Proposition 8 exist converge to modified paintboxes, which seems more difficult given the exceptional non-singleton sets of zero limiting frequency. On the other hand, our de Finetti argument only identifies the distribution of Π restricted to $\{n+1, n+2, \dots\}$ and gives little information about the conditional distribution of how the blocks of π attach themselves to such paintboxes. We have not found a simple and direct argument to see why the modified paintboxes describe the only way to attach π in an exchangeable way.

Now recall that Theorem 3 states that restricted exchangeable measures on \mathcal{P} are precisely those of the form $\mu = \sum_{\pi \in \mathcal{C}_{\mathcal{E}}} \kappa_{\mathbf{S}}^{\pi} \nu_{\pi}(ds)$.

Proof of Theorem 3. First consider $\mu = \sum_{\pi \in \mathcal{C}_{\mathcal{E}}} \kappa_{\mathbf{S}}^{\pi} \nu_{\pi}(ds)$. Take as \mathcal{P}^0 the pre-image under the bijection $b : \mathcal{P} \rightarrow \mathcal{G}$ of the set of infinite chains in \mathcal{E} . Then $\mu(\mathcal{P}^0) = 0$ because none of the measures $\int_{S^{\downarrow}} \kappa_{\mathbf{S}}^{\pi} \nu_{\pi}(ds)$ have mass in \mathcal{P}^0 . Indeed, the decomposition

$$\mathcal{P} = \mathcal{P}^0 \cup \bigcup_{\pi \in \mathcal{C}_{\mathcal{E}}} \mathcal{P}^{\pi}$$

is into disjoint measurable sets, and the restrictions of μ are finite and exchangeable on \mathcal{P}^{π} .

Conversely, let μ be any restricted exchangeable measure on \mathcal{P} and \mathcal{P}^j , $j \geq 0$, a measurable decomposition so that the restrictions of μ to \mathcal{P}^j are finite and exchangeable on \mathcal{P}^j . We first show that each of these restrictions of μ to \mathcal{P}^j is of the form required. Fix $j \geq 0$ and consider $\mathcal{P}_n^j = \{\pi \in \mathcal{P}_n : \mathcal{P}^{\pi} \subseteq \mathcal{P}^j\}$. Then we can define inductively

$$\mathcal{C}_1^j = \mathcal{P}_1^j, \quad \mathcal{C}_{n+1}^j = \mathcal{C}_n^j \cup \left(\mathcal{P}_{n+1}^j \setminus \bigcup_{\pi \in \mathcal{C}_n^j} \mathcal{K}^{\pi} \right), \quad n \geq 1, \quad \mathcal{C}^j = \bigcup_{n \geq 1} \mathcal{C}_n^j,$$

and obtain, by construction, disjoint \mathcal{P}^{π} , $\pi \in \mathcal{C}^j$, and exchangeable restrictions to these cylinder sets, which, by Proposition 9, can be represented as $\int_{S^{\downarrow}} \kappa_{\mathbf{S}}^{\pi} \nu_{\pi}(ds)$. Since furthermore $\mu(\{\Gamma \in \mathcal{P}^j : \Gamma|_n \not\subseteq \mathcal{P}_n^j \text{ for all } n \geq 1\}) = 0$, we have

$$\mu(\cdot \cap \mathcal{P}^j) = \sum_{\pi \in \mathcal{C}^j} \int_{S^{\downarrow}} \kappa_{\mathbf{S}}^{\pi} \nu_{\pi}(ds).$$

Now set $\mathcal{C} = \bigcup_{j \geq 0} \mathcal{C}^j$ and $\mathcal{E}_{\mathcal{C}} = \bigcup_{n \geq 0} \{\pi \cap [n] : \pi \in \mathcal{C} \cap \mathcal{P}_{n+1}\}$. If we now set \mathcal{E} to be the connected subgraph generated by $\mathcal{E}_{\mathcal{C}}$ and \emptyset , then \mathcal{E} differs from $\mathcal{E}_{\mathcal{C}}$ only by some $\pi' \preceq \pi \in \mathcal{E}_{\mathcal{C}}$ that do not contribute to $\mathcal{C}_{\mathcal{E}} = \mathcal{C}$. This completes the proof. \square

The proof of the Corollary 5 is now straightforward. The ν_j are not simply the ν_{π} for $\pi = \{[j], [j+1]\}$, $j \geq 1$, but almost. On the one hand, we have moved atoms of ν_{π} in $(0, 0, \dots)$ to a constant $k_j \geq 0$ and in $(1, 0, \dots)$ to a constant $c_j \geq 0$. The corresponding modified paintboxes are $\delta_{\{[j], [j+1], [j+2], \dots\}}$ and $\delta_{\{[j+1], \mathbb{N} \setminus [j+1]\}}$, respectively, except for $j = 1$, where the modified paintbox is $\frac{1}{2}(\delta_{\{\{1\}, \mathbb{N} \setminus \{1\}\}} + \delta_{\{\{2\}, \mathbb{N} \setminus \{2\}\}})$. On the other hand, we have incorporated the normalisation constants $Z_{\mathbf{S}}^{\pi}$ of the modified paintboxes as densities into ν_j and can and do here use restricted Kingman paintboxes $\kappa_{\mathbf{S}}(\cdot \cap \mathcal{P}^j)$ rather than normalised modified paintboxes $\kappa_{\mathbf{S}}^{\pi}$.

3 Basic results on restricted exchangeability and related notions

3.1 Partially exchangeable and constrained exchangeable partitions

Let us explore the connections between the restricted exchangeable partitions introduced in this paper and other generalisations of exchangeability studied in the literature, notably partial exchangeability and constrained exchangeability. *Partially exchangeable partitions* were introduced by Pitman [28]. A measure μ_n on \mathcal{P}_n is partially exchangeable if $\mu_n(\pi) = \mu_n(\pi')$ for all $\pi, \pi' \in \mathcal{P}_n$ with the same *vector* of block sizes *in the order of least element*. Partially exchangeable measures are not restricted exchangeable, in general, nor vice versa. Specifically, $\pi = (\{1, 2\}, \{3, 4\})$ and $\pi' = (\{1, 3\}, \{2, 4\})$ have the same mass for partially exchangeable measures but not necessarily for restricted exchangeable measures. Vice versa, consider $\pi = (\{1, 2, 3\}, \{4, 5\})$ and $\pi' = (\{1, 2\}, \{3, 4, 5\})$. In fact, “the intersection” of the two concepts is exchangeability:

Proposition 10 *A measure μ_n of \mathcal{P}_n is exchangeable if and only if it is both partially exchangeable and restricted exchangeable with decomposition $\mathcal{P}^1 = \mathcal{P}^{\{\{1\}, \{2\}\}}$, $\mathcal{P}^2 = \mathcal{P}^{\{\{2\}\}}$.*

Proof. The “only if” part follows straight from the definitions. For the “if” part, suppose that $\pi, \pi' \in \mathcal{P}_n \setminus \{\mathbf{1}_{[n]}\}$ have the same multiset of block sizes. Let $\tilde{\pi}$ be such that, for blocks in order of least element, $\tilde{\pi}_1 = (\pi_1 \cup \{\min \pi_2\}) \setminus \{2\}$ and $\tilde{\pi}_2 = (\pi_2 \setminus \{\min \pi_2\}) \cup \{2\}$, $\tilde{\pi}_j = \pi_j$, $j \geq 3$. Similarly construct $\tilde{\pi}'$ from π' . By partial exchangeability $\mu_n(\pi) = \mu_n(\tilde{\pi})$ and $\mu_n(\pi') = \mu_n(\tilde{\pi}')$. But $\tilde{\pi}', \tilde{\pi} \in \mathcal{P}^{\{\{1\}, \{2\}\}}$, so by restricted exchangeability, we have $\mu_n(\tilde{\pi}') = \mu_n(\tilde{\pi})$. \square

Constrained exchangeable partitions were introduced by Gnedin [15]. Let $\varsigma = (\varsigma_k, k \geq 1)$ be a fixed sequence of integers $\varsigma_k \geq 1$. Consider the set $\mathcal{P}^{\varsigma\text{-constr}}$ of partitions $\Gamma \in \mathcal{P}$ that are *constrained with respect to ς* in the sense that each block Γ_k contains the ς_k least elements of $\bigcup_{j \geq k} \Gamma_j$ for every $k \geq 1$ with $\Gamma_k \neq \emptyset$. A measure μ on \mathcal{P} is *constrained exchangeable* if $\mu(\mathcal{P} \setminus \mathcal{P}^{\varsigma\text{-constr}}) = 0$ for some ς , and if $\mu_n(\pi) = \mu_n(\pi')$ for all $\pi, \pi' \in \{\Gamma|_n : \Gamma \in \mathcal{P}^{\varsigma\text{-constr}}\}$ with the same multiset of block sizes and all $n \geq 1$. For $\varsigma = (1, 2, 1, \dots)$, under a constrained exchangeable measure, $\pi = (\{1, 3\}, \{2, 4\}, \{5\})$ and $\pi' = (\{1, 2\}, \{3, 4\}, \{5\})$ have the same mass, but not necessarily under a restricted exchangeable measure. Vice versa, restrictions to $\mathcal{P}^{\{\{j\}, \{j+1\}\}}$ of a restricted exchangeable measure μ are constrained exchangeable if we take $\varsigma = (j, 1, 1, \dots)$, but as soon as μ gives positive mass to more than one $\mathcal{P}^{\{\{j\}, \{j+1\}\}}$, $j \geq 1$, constrained exchangeability in Gnedin’s sense fails.

3.2 Hierarchies, ordered hierarchies, fragmentations and coalescents

Following [31, 33, 22], we call *hierarchy of $B \subseteq \mathbb{N}$* any subset \mathcal{H}_B of the power set of B such that $B \in \mathcal{H}_B$ and $\{k\} \in \mathcal{H}_B$ for all $k \in B$, and so that for every $A, A' \in \mathcal{H}_B$, either $A \subseteq A'$ or $A' \subseteq A$. To avoid trivialities, we also require $\emptyset \in \mathcal{H}_B$. We say that a hierarchy is closed if for all sequences $(A_n, n \geq 1)$ in \mathcal{H} that are increasing for the inclusion partial order, we have $\bigcup A_n \in \mathcal{H}_B$, and if for all decreasing sequences $\bigcap A_n \in \mathcal{H}_B$. We say that a strict subset $A \subset A'$ is *maximal in \mathcal{H}_B* if for all $A'' \in \mathcal{H}_B$ with $A \subseteq A'' \subseteq A'$ either $A = A''$ or $A'' = A'$. For finite $B \subset \mathbb{N}$, the maximal subsets A_1, \dots, A_k of B in \mathcal{H}_B form a partition of B and the restrictions $\mathcal{H}_{A_i} = \mathcal{H}_B \cap A_i = \{A \cap A_i : A \in \mathcal{H}_B\}$ are hierarchies of A_i , $i \in [k]$. This is not always true for infinite $B \subseteq \mathbb{N}$ and so it will be useful to note that a closed hierarchy \mathcal{H}_B is uniquely determined by its restrictions $\mathcal{H}_B \cap [n]$, $n \geq 1$, as $\mathcal{H}_B = \{A \subseteq B : A \cap [n] \in \mathcal{H}_B \cap [n] \text{ for all } n \geq 1\}$.

We say that a family $(Q_n, n \geq 2)$ of distributions on the set of hierarchies of $[n]$, $n \geq 2$, is *consistent* if for $T_{n+1} \sim Q_{n+1}$ we have $T_{n+1} \cap [n] \sim Q_n$. Let $T_n \sim Q_n$. We introduce the partition Π_n into maximal strict subsets of $[n]$ and refer to its distribution P_n on \mathcal{P}_n as *splitting rule*. We say that $(Q_n, n \geq 2)$ is a *labelled Markov branching model* if conditionally given $\Pi_n = \pi$, the

hierarchies $T_n \cap \pi_i$, $i \geq 1$, have as their distribution $Q_{\#\pi_i}$ pushed forward under the natural bijection on the set of hierarchies induced by the increasing bijection from $[\#\pi_i]$ to π_i .

Let $\mathcal{P}^j = \mathcal{P}^{\{[j], \{j+1\}\}}$, $j \geq 1$. We say that a splitting rule P_n is *restricted exchangeable* if for all $1 \leq j \leq n-1$ and $\pi, \pi' \in \mathcal{P}_n^j$, we have $P_n(\{\pi\}) = P_n(\{\pi'\})$. Clearly, if κ is a restricted exchangeable dislocation measure as in Corollary 5, then

$$P_n(\{\pi\}) = \kappa(\mathcal{P}^\pi) / \kappa(\mathcal{P} \setminus \mathcal{P}^{\mathbf{1}_{[n]}}), \quad \pi \in \mathcal{P}_n \setminus \{\mathbf{1}_{[n]}\}, \quad n \geq 2, \quad (4)$$

defines restricted exchangeable splitting rules and hence inductively a consistent Markov branching model $(Q_n, n \geq 2)$. The converse also holds:

Proposition 11 *All consistent labelled Markov branching models $(Q_n, n \geq 2)$ with restricted exchangeable splitting rules $(P_n, n \geq 2)$ are of the form (4) for some restricted exchangeable measure κ as in Corollary 5.*

Proof. In Pitman's [29] formalism of exchangeable partition probability functions (EPPFs)

$$p_n^j(\#\pi_1, \dots, \#\pi_k) = P_n(\{\pi\}) \quad \pi \in \mathcal{P}_n^j = \mathcal{P}^{\{[j], \{j+1\}\}} \cap [n], \quad 1 \leq j \leq n-1,$$

consistency in the restricted exchangeable case (extending [26, Formula (16)]) is equivalent to

$$p_n^j(n_1, \dots, n_k) = p_{n+1}^n(n, 1) p_n^j(n_1, \dots, n_k) + \sum_{i=1}^{k+1} p_{n+1}^j(n_1, \dots, n_{i-1}, n_i + 1, n_{i+1}, \dots, n_k).$$

For any $\lambda_2 \in (0, \infty)$ and $(1 - p_{n+1}^n(n, 1))\lambda_{n+1} = \lambda_n$, $n \geq 2$, we see that $\kappa(\mathcal{P}^\pi) = \lambda_n P_n(\{\pi\})$, $\pi \in \mathcal{K}$, defines a restricted exchangeable measure that has the properties required. \square

By Kolmogorov's consistency theorem, we can consider a consistent family $(T_n, n \geq 1)$ of trees $T_n \sim Q_n$ with $T_{n+1} \cap [n] = T_n$, $n \geq 1$, and associate $\mathcal{H} = \{A \subset \mathbb{N} : A \cap [n] \in T_n \text{ for all } n \geq 2\}$ as random hierarchy of \mathbb{N} . In the setting of Corollary 5, we can consistently embed into continuous time the blocks of T_n , $n \geq 2$, using exponential holding times $\eta_{[n]}$ of rate λ_n for state $[n]$ and then recursively, η_{π_i} at rate $\lambda_{\#\pi_i}$ for any maximal strict subset π_i that is created when $[n]$ splits, to construct consistent homogeneous fragmentation processes $(F|_n(t), t \geq 0)$ in \mathcal{P}_n , $n \geq 2$. We can also generalise Bertoin's [8] Poissonian construction to directly obtain restricted exchangeable homogeneous fragmentation processes $(F(t), t \geq 0)$ in \mathcal{P} , that provide an alternative construction of the same random hierarchy $\mathcal{H} = \{F_i(t), i \geq 0, t \geq 0\}^{\text{cl}}$, but we do not need this alternative construction and leave the details to the reader. We do note, however that this construction also provides an ordered hierarchy $\mathcal{H}^{\text{ord}} = \{F(t), t \geq 0\}^{\text{cl}} \subset \mathcal{P}$ of \mathbb{N} . For any such construction, based on κ , say, to be meaningful, we require $\lambda_n = \kappa(\mathcal{P} \setminus \mathcal{P}^{\mathbf{1}_{[n]}}) < \infty$ for all $n \geq 2$.

For $B \subseteq \mathbb{N}$, we call *ordered hierarchy of B* any subset $\mathcal{H}^{\text{ord}} \subseteq \mathcal{P}_B$ with $\mathbf{1}_B \in \mathcal{H}^{\text{ord}}$ and $\mathbf{0}_B \in \mathcal{H}^{\text{ord}}$, such that for all $\Gamma, \Gamma' \in \mathcal{H}^{\text{ord}}$, we have $\Gamma \preceq \Gamma'$ or $\Gamma' \preceq \Gamma$, where $\Gamma \preceq \Gamma'$ means that for every block Γ_i of Γ there is a block Γ'_j of Γ' with $\Gamma_i \subseteq \Gamma'_j$.

We say that a family $(Q_n, n \geq 2)$ of distributions on the set of ordered hierarchies of $[n]$, $n \geq 2$, is consistent if for $\mathcal{H}_{n+1}^{\text{ord}} \sim Q_{n+1}$, we have $\mathcal{H}_{n+1}^{\text{ord}} \cap [n] \sim Q_n$. Let $\mathcal{H}_n^{\text{ord}} \sim Q_n$. We introduce the partition $\Pi_n \in \mathcal{H}_n^{\text{ord}} \setminus \{\mathbf{0}_{[n]}\}$ that is minimal for the partial order \preceq and refer to its distribution P_n on \mathcal{P}_n as *merging rule*. We say that $(Q_n, n \geq 2)$ is a discrete coalescent model if conditionally given $\Pi_n = \pi = (\pi_1, \dots, \pi_k)$, the ordered hierarchy $\mathcal{H}_n^{\text{ord}} \cap \{\min \pi_i, i \in [k]\}$ has as its distribution Q_k pushed forward under the natural bijection on the set of ordered hierarchies induced by the increasing bijection from $[k]$ to $\{\min \pi_i, i \in [k]\}$.

For $j = (k-1)(k-2)/2 + i$, let $\mathcal{P}^j = \mathcal{P}^{\{\mathbf{0}_{[k-1]} \setminus \{i\}\} \cup \{\{i, k\}\}}$, $1 \leq i < k$. A merging rule P_n is *restricted exchangeable* if for all $1 \leq i < k \leq n$ and $\pi, \pi' \in \mathcal{P}_n^{ik}$, we have $P_n(\{\pi\}) = P_n(\{\pi'\})$.

Proposition 12 *All consistent discrete coalescent models $(Q_n, n \geq 2)$ with restricted exchangeable merging rules $(P_n, n \geq 2)$ are of the form*

$$P_n(\{\pi\}) = L(\mathcal{P}^\pi)/L(\mathcal{P} \setminus \mathcal{P}^{0_{[n]}}), \quad \pi \in \mathcal{P}_n \setminus \{0_{[n]}\}, n \geq 2, \quad (5)$$

where L is a restricted exchangeable measure on \mathcal{P} with decomposition $\mathcal{P}^0 = \{0_{\mathbb{N}}\}$ and $\mathcal{P}^j, j \geq 1$.

Generalising Schweinsberg [32], we call such measures L on \mathcal{P} *coalescent L -measures*.

Proof. The proof is similar to Proposition 11. In obvious notation, consistency is equivalent

$$p_n^{ik}(n_1, \dots, n_\ell) = \left(\sum_{j=1}^n p_{n+1}^{j, n+1}(1, \dots, 1, 2) \right) p_n^{ik}(n_1, \dots, n_\ell) + \sum_{j=1}^{\ell+1} p_{n+1}^{ik}(n_1, \dots, n_j + 1, \dots, n_\ell),$$

and we need $\left(1 - \sum_{j=1}^n p_{n+1}^{j, n+1}(1, \dots, 1, 2)\right) \lambda_{n+1} = \lambda_n, n \geq 2$, to conclude. \square

In fact, an analogous characterisation result also holds for any other decomposition of \mathcal{P} into any zero-mass \mathcal{P}^0 and cylinder sets $\mathcal{P}^j, j \geq 1$, or unions of cylinder sets.

To consistent $\mathcal{H}_n^{\text{ord}}, n \geq 2$, we associate $\mathcal{H}^{\text{ord}} = \{\Gamma \in \mathcal{P} : \Gamma \cap [n] \in \mathcal{H}_n^{\text{ord}} \text{ for all } n \geq 2\}$ as random ordered hierarchy of \mathbb{N} and also processes $(F|_n(t), t \geq 0)$ and $(F(t), t \geq 0)$ by embedding into continuous time using rates $\lambda_n, n \geq 2$, as well as Schweinsberg's [32] Poissonian construction.

3.3 Integral representation for restricted exchangeable coalescent L -measures

As a direct consequence of Theorem 3, mimicking Corollary 5, we obtain

Proposition 13 *Let L be a restricted exchangeable coalescent L -measure. Then there are constants $c_{ij} \geq 0, k_{ij} \geq 0$ and a measure ν_{ij} on \mathcal{S}^\downarrow satisfying*

$$\int_{\mathcal{S}^\downarrow} \left(\sum_{i_0, \dots, i_{j-2}} s_{i_0}^2 \prod_{m=1}^{j-2} s_{i_m} \right) \nu_{ij}(ds) < \infty, \quad 1 \leq i < j,$$

where the sum is over indices that are distinct or zero except that $i_0 \neq 0$, such that

$$L = \sum_{1 \leq i < j < \infty} \left(c_{ij} \delta_{\omega\{i, j\}} + k_{ij} \delta_{\varepsilon[j] \setminus \{i\}} + \int_{\mathcal{S}^\downarrow} \kappa_{\mathbf{s}}(\cdot \cap \mathcal{P}^{ij}) \nu_{ij}(ds) \right),$$

where $\kappa_{\mathbf{s}}$ is Kingman's paintbox.

4 Embedding in self-similar CRTs, proof of Theorem 6

4.1 Self-similar CRTs, fragmentation processes and spinal decomposition

Aldous [2] called a pair (\mathcal{T}, μ) a *continuum tree* if \mathcal{T} is an \mathbb{R} -tree, μ a finite measure on \mathcal{T} , with

1. the measure μ supported by the set $\text{Lf}(\mathcal{T})$ of leaves of \mathcal{T} ,
2. the measure μ atomless,
3. for every $x \in \mathcal{T} \setminus \text{Lf}(\mathcal{T})$, positive mass $\mu(\mathcal{T}_x) > 0$ in the subtree \mathcal{T}_x rooted at x .

In the sequel, we will often specify a root vertex $\rho \in \mathcal{T}$ and the distance function d . For technical simplicity, we follow Aldous [3] and use CRTs in $\ell_1 = \ell_1(\mathbb{N})$. We endow the set of compact subsets of ℓ_1 with the Hausdorff metric, and the set of finite measures on ℓ_1 with any metric inducing the topology of weak convergence, so that the set \mathbb{H} of pairs (T, μ) where T is a rooted \mathbb{R} -tree embedded as a subset of ℓ_1 and μ is a finite measure on T , is endowed with the product Borel σ -algebra.

A *Continuum Random Tree (CRT)* is a random variable with values in the set of continuum trees. To be specific, we call *distribution of a CRT* $(\mathcal{T}, \mu, \rho, d)$ the distribution on \mathbb{H} of the particular random isometric embedding in ℓ_1 obtained from a random sample Σ_i^* , $i \geq 1$, of independent leaves with distribution $\mu/\mu(\mathcal{T})$, using $0 \in \ell_1$ as the root and the i th coordinate direction in ℓ_1 to embed the branch leading to leaf Σ_i^* , finally passing to the ℓ_1 -closure and the weak limit of the $\mu(\mathcal{T})$ -multiples of empirical measures of the embedded $\Sigma_1^*, \dots, \Sigma_i^*, i \geq 1$.

For $x \in [0, 1]$ and $\mathbf{s} \in S^\downarrow$, we denote by Q_x^α the distribution of the α -scaled tree $(\mathcal{T}, x\mu, \rho, x^\alpha d)$ and by $Q_{\mathbf{s}}^\alpha$ the distribution of a bush of independent trees with distributions $Q_{s_i}^\alpha$, $i \geq 1$, all grafted to the same root. For every $u \geq 0$, consider the bush $\mathcal{B}^\alpha(u)$ obtained by grafting the connected components $\mathcal{T}_i^\alpha(u)$, $i \in I$, of the open set $\{x \in \mathcal{T} : d(x, \rho) > u\}$ to the same root. A CRT is called α -self-similar in the sense of [19], if for all $u \geq 0$ and conditionally given $(\mu(\mathcal{T}_i^\alpha(u)), i \in I)^\downarrow = \mathbf{s} \neq 0$, we have $\mathcal{B}^\alpha(u) \sim Q_{\mathbf{s}}^\alpha$.

For $\alpha \in \mathbb{R}$, a \mathcal{P} -valued process $(\Pi(t), t \geq 0)$ is an *exchangeable α -self-similar fragmentation process* if given $\Pi(t) = \pi$, the partition $\Pi(t+s)$ has the same law as the random partition whose blocks are those of $\pi_i \cap \Pi^{(i)}(|\pi_i|^\alpha s)$, $i \geq 1$, where $(\Pi^{(i)}, i \geq 1)$ is a sequence of i.i.d. copies of $(\Pi(t), t \geq 0)$. The process $X^\alpha = (|\Pi(t)|^\downarrow, t \geq 0)$ is an S^\downarrow -valued α -self-similar fragmentation. Bertoin proved in [5] that the distribution of an exchangeable \mathcal{P} -valued self-similar fragmentation is determined by a triple (α, c, ν) , where ν is a dislocation measure on S^\downarrow , i.e. $\nu(s_1 = 1) = 0$ and $\int_{S^\downarrow} (1 - s_1) \nu(ds) < \infty$. For this paper, we are only interested in the case $c = 0$ and when ν is *conservative*, i.e. $\nu(s_0 > 0) = 0$, where $s_0 = 1 - \sum_{i \geq 1} s_i$. We call (α, ν) characteristic pair.

Haas and Miermont in [19] have shown that there exists a self-similar continuum random tree characterized by such a pair (α, ν) , provided also that ν is infinite. Specifically, $Y^\alpha = ((\mu(\mathcal{T}_i^\alpha(t)), i \in I)^\downarrow, t \geq 0)$ has the same distribution as X^α .

Consider $\mathbf{s} \in S^\downarrow$ and $\mathbf{s}^{(i)} \in S^\downarrow$, $i \geq 1$. We call *fragmentation of \mathbf{s} by $\mathbf{s}^{(\cdot)}$* the mass partition $\text{Frag}(\mathbf{s}, \mathbf{s}^{(\cdot)})$ given by the decreasing rearrangement of $(s_i s_j^{(i)}, i, j \in \mathbb{N})$. Bertoin showed that the process $(X^\alpha(t), t \geq 0)$ is Markovian and its semigroup can be described as follows. For every $t, t' \geq 0$, the conditional distribution of $X^\alpha(t+t')$ given $X^\alpha(t) = \mathbf{s}$ is the law of $\text{Frag}(\mathbf{s}, \mathbf{S}^{(\cdot)})$, where each $\mathbf{S}^{(i)}$ independently is distributed as $X^\alpha(t' s_i^{-\alpha})$, see [8, Proposition 3.7].

Given a CRT $(\mathcal{T}, \mu, \rho, d)$ and $v \in \mathcal{T}$, we denote by $v^\alpha(u)$ the point on the *spine* $[[\rho, v]]$ with $d(\rho, v^\alpha(u)) = u$ and obtain a parameterisation by distance $[[\rho, v]] = \{v^\alpha(u), 0 \leq u \leq d(\rho, v)\}$. We consider the subtree $\mathcal{T}_{(v)}^\alpha(u) = \{w \in \mathcal{T} : d(\rho, w \wedge v) > u\}$ of \mathcal{T} containing v rooted at $v^\alpha(u)$, and its mass $X_{(v)}^\alpha(u) = \mu(\mathcal{T}_{(v)}^\alpha(u))$. Let η_v be the self-similar time change with

$$\eta_v(t) = \inf \left\{ u \geq 0 : \int_0^u (X_{(v)}^\alpha(y))^{-\alpha} dy > t \right\}, \quad 0 \leq t < \zeta_v = \int_0^{d(\rho, v)} (X_{(v)}^\alpha(y))^{-\alpha} dy. \quad (6)$$

Then $v^0(t) = v^\alpha(\eta_v(t))$, $\mathcal{T}_{(v)}^0(t) = \mathcal{T}_{(v)}^\alpha(\eta_v(t))$ and $X_{(v)}(t) = \mu(\mathcal{T}_{(v)}^0(t))$ are the associated quantities in homogeneous time. Note, in particular, the parameterisation of the spine in homogeneous time $[[\rho, v]] = \{v^0(t), 0 \leq t < \zeta_v\}$. Denote by $S^v(t) = (S_i^v(t), i \geq 1) \in S^\downarrow$ the sequence such that $X_{(v)}(t-) S^v(t)$ is the decreasing sequence of μ -masses of the connected components of $\{w \in \mathcal{T} : v^0(t) \in [[\rho, w]]\}$, also $F_v(t) = X_{(v)}(t)/X_{(v)}(t-)$ the member of $S^v(t)$ corresponding to the subtree containing v . Moreover, we denote by

$$\left(\mathcal{B}_{(v)}^0(t), \frac{\mu|_{\mathcal{B}_{(v)}^0(t)}}{X_{(v)}(t-)}, v^0(t), \frac{d|_{\mathcal{B}_{(v)}^0(t)}}{(X_{(v)}(t-))^\alpha} \right), \quad \text{where } \mathcal{B}_{(v)}^0(t) = \mathcal{T}_{(v)}^0(t-) \setminus \mathcal{T}_{(v)}^0(t)$$

the associated rescaled spinal bush, of mass $1 - F_\nu(t)$, at homogeneous time $t \geq 0$.

The following lemma is a description in the CRT framework of Bertoin's tagged particle process that is a bit richer than often stated, but follows from the same arguments.

Lemma 14 *Let $(\mathcal{T}, \mu, \rho, d)$ be an α -self-similar CRT with characteristic pair (α, ν) and $\Sigma^* \sim \mu$. Then $(S^{\Sigma^*}, F_{\Sigma^*})$ is a Poisson point process on $S^\downarrow \times (0, 1)$ with intensity measure $\tilde{\nu}^*$ given by*

$$\tilde{\nu}^*(ds \otimes dx) = \sum_{i=1}^{\infty} s_i \delta_{s_i}(dx) \nu(ds).$$

Proof. Let X^α be the self-similar mass-fragmentation process corresponding to the CRT (\mathcal{T}, μ) and X the homogeneous mass-fragmentation process of X^α through the self-similar time-change. Extending the probability space, if necessary, denote by Π a homogeneous exchangeable \mathcal{P} -valued fragmentation process associated with X . Without loss of generality, we can consider $X_{(\Sigma^*)}(t) = |\Pi_1(t)|$ by exchangeability. Let $\Pi^{(1)}(t)$ be the partition of \mathbb{N} such that $\Pi_1(t) = \text{Frag}(\Pi_1(t-), \Pi^{(1)}(t))$, then $S^{\Sigma^*}(t) = |\Pi^{(1)}(t)|^\downarrow$. By the Poissonian construction of exchangeable fragmentation processes, $\Pi^{(1)}$ is a Poisson point process with intensity measure $\kappa = \int_{S^\downarrow} \kappa_s \nu(ds)$. Hence, S^{Σ^*} is a Poisson point process on S^\downarrow with intensity measure ν .

As Σ^* is chosen according to μ , it is not hard to show that $(S^{\Sigma^*}, F_{\Sigma^*})$ relates to S^{Σ^*} via the size-biased marking kernel $K^*(s, \cdot) = \sum_{i \geq 1} s_i \delta_{s_i}$ and so $(S^{\Sigma^*}, F_{\Sigma^*})$ is a Poisson point process with intensity $K^*(s, dx) \nu(ds) = \tilde{\nu}^*(ds \otimes dx)$. \square

By the stopping line argument of [21, Proposition 4], this yields the following joint description of the ordered coarse and unordered fine spinal decompositions along the spine to $\Sigma^* \sim \mu$.

Proposition 15 (Spinal decomposition [9, 21]) *Let $(\mathcal{T}, \mu, \rho, d)$ be an α -self-similar CRT with characteristic pair (α, ν) and $\Sigma^* \sim \mu$. Then the process $(S^{\Sigma^*}, F_{\Sigma^*}, \mathcal{B}_{(\Sigma^*)}^0)$ is a Poisson point process with intensity measure*

$$\tilde{\nu}_{\text{bush}}^*(ds \otimes dx \otimes dT) = \sum_{i \geq 1} s_i \delta_{s_i}(dx) Q_{(s_1, \dots, s_{i-1}, s_{i+1}, \dots)}(dT) \nu(ds).$$

Conversely, $(\mathcal{T}, \mu, \rho, d)$ is a measurable function of $(S^{\Sigma^}, F_{\Sigma^*}, \mathcal{B}_{(\Sigma^*)}^0)$*

4.2 A generic procedure to sample a leaf from a self-similar CRT

Our aim is to generalise Lemma 14 and Proposition 15 to leaves other than the μ -sampled leaf Σ^* where we are effectively marking a Poisson point process with intensity measure ν using the size-biased marking kernel $K^*(s, \cdot) = \sum_{i \geq 1} s_i \delta_{s_i}$ from S^\downarrow to $(0, 1)$. We will now consider other marking kernels. It will be convenient to adopt an idea from Pitman's EPPF formalism and specify the probability that a *specific* part of size x is chosen with probability $P(s, x)$ so that the probability of choosing a mass x is $K(s, \{x\}) = m_x P(s, x)$ where m_x is the number of $i \geq 1$ with $s_i = x$ in $\mathbf{s} = (s_i, i \geq 1) \in S^\downarrow$.

Definition 16 A measurable function $P : S^\downarrow \times (0, 1) \rightarrow [0, 1]$ that fulfils the two conditions

- $P(\mathbf{s}, x) = 0$ if $x \notin \{s_i, i \geq 1\}$;
- $\sum_{i \geq 1} P(\mathbf{s}, s_i) = 1$.

is called a selection probability function (SPF).

Example 17 The SPF associated with a leaf chosen according to μ is $P(\mathbf{s}, s_i) = s_i$.

Let us now explain a generic procedure to sample a single special leaf Σ based on an SPF P from a self-similar CRT $(\mathcal{T}, \mu, \rho, d)$ with dislocation measure ν .

Procedure 1 Let P be an SPF as in Definition 16 fulfilling

$$\int_{S^1} \sum_{i=1}^{\infty} (1 - s_i) P(\mathbf{s}, s_i) \nu(d\mathbf{s}) < \infty. \quad (7)$$

0. We start from $(\mathcal{T}_1, \mu_1, \rho_1, d_1) := (\mathcal{T}, \mu, \rho, d)$ and $i = 1$.
1. Conditionally given $(\mathcal{T}_i, \mu_i, \rho_i, d_i)$, let $\Sigma_{(i)} \sim \mu_i$.
2. Conditionally given $(\mathcal{T}_i, \Sigma_{(i)})$, we consider the parameterisation in homogenous time of the spine $[[\rho_i, \Sigma_{(i)}] = \{\Sigma_{(i)}^0(t), t \geq 0\}$ and pick as \mathcal{T}_{i+1} a subtree \mathcal{S} above $\Sigma_{(i)}^0(t)$ with probability

$$\mathbb{P}(\mathcal{T}_{i+1} = \mathcal{S} | \mathcal{T}_i, \Sigma_{(i)}) = P \left(S^{\Sigma_{(i)}}(t), \frac{\mu_i(\mathcal{S})}{\mu_i(\mathcal{T}_{(\Sigma_{(i)}^0(t-))})} \right) \prod_{t' < t} P \left(S^{\Sigma_{(i)}}(t'), F_{\Sigma_{(i)}}(t') \right).$$

3. Let $\tau_{(i)} = \inf\{t \geq 0 : \mathcal{T}_{i+1} \cap \mathcal{T}_{\Sigma_{(i)}^0}(t) = \emptyset\}$. We turn \mathcal{T}_{i+1} into a CRT with rescaled mass measure, root and rescaled distance function as follows:

$$\mu_{i+1} = \frac{\mu_i|_{\mathcal{T}_{i+1}}}{\mu_i(\mathcal{T}_{i+1})}, \quad \rho_{i+1} = \Sigma_{(i)}^0(\tau_{(i)}), \quad d_{i+1} = \frac{d_i|_{\mathcal{T}_{i+1} \times \mathcal{T}_{i+1}}}{(\mu_i(\mathcal{T}_{i+1}))^\alpha}.$$

4. Repeat within the subtree $(\mathcal{T}_{i+1}, \mu_{i+1}, \rho_{i+1}, d_{i+1})$ by increasing i by 1 and proceeding to 1.
5. As $i \rightarrow \infty$, we obtain a sequence $(\Sigma_{(i)}^0(\tau_{(i)}), i \geq 1)$ in \mathcal{T} that increases in the sense that $\Sigma_{(i)}^0(\tau_{(i)}) \in [[\rho, \Sigma_{(i+1)}^0(\tau_{(i+1)})]]$ and hence converges. Let $\Sigma = \lim_{i \rightarrow \infty} \Sigma_{(i)}^0(\tau_{(i)})$.

Roughly speaking, this sampling procedure is that we travel along the spine $[[\rho, \Sigma_{(1)}]]$ and keep selecting subtrees until the first time we choose a subtree not containing $\Sigma_{(1)}$ and then repeat inductively in the subtree until we reach a leaf Σ in the limit. We show in the following proposition that there is a spinal subordinator associated with Σ .

Proposition 18 *Let Σ be sampled according to Procedure 1.*

- (i) *Then the process $(S^\Sigma, F_\Sigma, \mathcal{B}_{(\Sigma)}^0)$ is a Poisson point process with intensity measure*

$$\tilde{\nu}_{\text{bush}}^P(ds \otimes dx \otimes dT) = \sum_{i \geq 1} P(\mathbf{s}, s_i) \delta_{s_i}(dx) Q_{(s_1, \dots, s_{i-1}, s_{i+1}, \dots)}(dT) \nu(d\mathbf{s}).$$

Specifically, $\left((S^\Sigma(t), F_\Sigma(t), \mathcal{B}_{(\Sigma)}^0(t)), 0 \leq t < \tau_{(1)} \right)$ is a killed Poisson point process with killing rate $\int_{S^1} \sum_{i=1}^{\infty} (1 - s_i) P(\mathbf{s}, s_i) \nu(d\mathbf{s})$ and (accordingly thinned) intensity measure

$$\tilde{\nu}_{(1), \text{bush}}^P(ds \otimes dx \otimes dT) = \sum_{i=1}^{\infty} s_i P(\mathbf{s}, s_i) \delta_{s_i}(dx) Q_{(s_1, \dots, s_{i-1}, s_{i+1}, \dots)}(dT) \nu(d\mathbf{s}).$$

- (ii) *Let $\xi_t^\Sigma = -\log X_{(\Sigma)}(t)$, $t \geq 0$. Then ξ^Σ is a pure jump subordinator with Laplace exponent Φ_Σ and Lévy measure Λ_Σ given by*

$$\Phi_\Sigma(q) = \int_{S^1} \sum_{i=1}^{\infty} (1 - s_i^q) P(\mathbf{s}, s_i) \nu(d\mathbf{s}) \quad \text{and} \quad \Lambda_\Sigma = \sum_{i=1}^{\infty} P(\mathbf{s}, s_i) \delta_{-\log s_i} \nu(d\mathbf{s}). \quad (8)$$

Proof. (i) This proof relies heavily on Poisson point process techniques. We use the terminology of Kingman [25]. By Proposition 15, the process $(S^{\Sigma(1)}, F_{\Sigma(1)}, \mathcal{B}_{(\Sigma(1))}^0)$ is a Poisson point process with intensity measure $\tilde{\nu}_{\text{bush}}^*$. Step 2. of Procedure 1 can be read and analysed as follows. We mark some points of this Poisson point process with a selected subtree $\mathcal{T}_{(\Sigma(1))}^{\text{sel}}(t)$ using the kernel

$$K(\mathbf{s}, x, B'; dT'') = P(\mathbf{s}, x)\delta_{\{\emptyset\}}(dT'') + \sum_{S \text{ connected component of } B' \setminus \{\rho'\}} P(\mathbf{s}, \mu'(S))\delta_S(dT''),$$

where B' is short for (B', μ', ρ', d') and T'' is short for $(T'', \mu'', \rho'', d'')$, also S for $(S, \mu'|_S, \rho', d'|_{S \times S})$ and $\{\emptyset\}$ for $(\{\emptyset\}, 0, 0, 0)$. By standard marking and mapping, we get a new Poisson point process $(S^{\Sigma(1)}, F_{\Sigma(1)}, \mathcal{B}_{(\Sigma(1))}^{\text{rem}}, \mathcal{T}_{(\Sigma(1))}^{\text{sel}})$, where $\mathcal{B}_{(\Sigma(1))}^{\text{rem}}(t) = \mathcal{B}_{(\Sigma(1))}^0(t) \setminus \mathcal{T}_{(\Sigma(1))}^{\text{sel}}(t)$ with intensity measure

$$\sum_{i \geq 1} s_i \delta_{s_i}(dx) \left(P(\mathbf{s}, s_i) Q_{\widehat{\mathbf{s}}^{(i)}}(dB') \delta_{\{\emptyset\}}(dT'') + \sum_{j \neq i} P(\mathbf{s}, s_j) Q_{\widehat{\mathbf{s}}^{(i,j)}}(dB') Q_{s_j}(dT'') \right) \nu(ds),$$

where $\widehat{\mathbf{s}}^{(i)} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots)$ is the sequence \mathbf{s} with s_i removed and similarly $\widehat{\mathbf{s}}^{(i,j)}$ is the sequence \mathbf{s} with s_i and s_j removed.

In Step 3., we set $\tau_{(1)} = \inf\{t \geq 0 : \mathcal{T}_{(\Sigma(1))}^{\text{sel}}(t) \neq \{\emptyset\}\}$, exponentially distributed with rate

$$\int_{S^{\downarrow}} \sum_{i \geq 1} s_i \sum_{j \neq i} P(\mathbf{s}, s_j) \nu(ds) = \int_{S^{\downarrow}} \sum_{j \geq 1} (1 - s_j) P(\mathbf{s}, s_j) \nu(ds) < \infty,$$

note (7). Standard thinning and projecting yields that $((S^{\Sigma}(t), F_{\Sigma}(t), \mathcal{B}_{(\Sigma)}^0(t)), 0 \leq t < \tau_{(1)}) = ((S^{\Sigma(1)}(t), F_{\Sigma(1)}(t), \mathcal{B}_{(\Sigma(1))}^{\text{rem}}(t)), 0 \leq t < \tau_{(1)})$ is an independently killed Poisson point process with intensity measure $\sum_{i \geq 1} s_i P(\mathbf{s}, s_i) \delta_{s_i}(dx) Q_{\widehat{\mathbf{s}}^{(i)}}(dB') \nu(ds)$, as required for the second assertion. The rescaled tree $\mathcal{T}_2 = \mathcal{T}_{(\Sigma(1))}^{\text{sel}}(\tau_{(1)}) \sim Q_1$ is independent of this killed Poisson point process and also jointly independent of the pair formed by the bush $\mathcal{B}_{(\Sigma(1))}^{\text{rem}}$ and the rescaled tree $\mathcal{T}_{(\Sigma(1))}^0(\tau_{(1)})$ that has distribution Q_{s_i} for $s_i = F_{\Sigma(1)}(\tau_{(1)})$, using the converse statement in Proposition 15.

In Step 4., the induction proceeds on $\mathcal{T}_i \sim Q_1$, $i \geq 2$, all independent of the past, so this Poisson point process extends indefinitely, but ignores points at $\tau_{(1)} + \dots + \tau_{(i)}$, $i \geq 1$. These are exponentially spaced and i.i.d., hence form an independent Poisson point process. The independence and distributional properties that we noted identify the distribution of $(S^{\Sigma}(\tau_{(1)}), F_{\Sigma}(\tau_{(1)}), \mathcal{B}_{(\Sigma)}^0(\tau_{(1)})) = (S^{\Sigma(1)}(\tau_{(1)}), \mu^{\text{sel}}(\mathcal{T}_{(\Sigma(1))}^{\text{sel}}(\tau_{(1)})), \widetilde{\mathcal{B}}_1)$, and so the intensity measure

$$\sum_{i \geq 1} s_i \sum_{j \neq i} P(\mathbf{s}, s_j) \delta_{s_j}(dx) Q_{(s_i, \widehat{\mathbf{s}}^{(i,j)})}(dB') \nu(ds) = \sum_{j \geq 1} (1 - s_j) P(\mathbf{s}, s_j) \delta_{s_j}(dx) Q_{\widehat{\mathbf{s}}^{(j)}}(dB') \nu(ds),$$

because $\widetilde{\mathcal{B}}_i$ is obtained by grafting to the same root $\mathcal{B}_{(\Sigma(i))}^{\text{rem}}(\tau_{(i)}) \sim Q_{\widehat{\mathbf{s}}^{(i,j)}}$ and the rescaled $\mathcal{T}_{(\Sigma(i))}^0(\tau_{(i)})$ with distribution Q_{s_i} . Standard superposition completes the proof of (i).

(ii) By (i) and standard mapping, $(\Delta \xi_t^{\Sigma}, t \geq 0)$ is a Poisson point process with intensity measure Λ_{Σ} , hence $\xi_t^{\Sigma} = \sum_{s \leq t} \Delta \xi_s^{\Sigma}$ is a pure jump subordinator with Laplace exponent Φ_{Σ} . \square

Remark 19 In fact, Proposition 18 (ii) shows that for any Lévy measure Λ with the form in (8), we can find a leaf Σ from this generic sampling procedure with some selection probability P such that the Lévy measure of its spinal subordinator coincides with Λ .

4.3 A procedure to sample a sequence of leaves from a self-similar CRT

In this section, we formulate a special inductive procedure to sample k leaves $\Sigma_1, \dots, \Sigma_k$ from a self-similar CRT (\mathcal{T}, μ) with characteristic pair (α, ν) , where

$$\nu(ds) = \sum_{j \geq 1} \left(\sum_{i \geq 1} s_i^j (1 - s_i) \right) \nu_j(ds)$$

for some finite measures ν_j , $j \geq 1$, representing a restricted exchangeable dislocation measure as in Corollary 5. Clearly, the measures ν_j , $j \geq 1$, are absolutely continuous with respect to ν . We denote their Radon-Nikodym derivatives by $f_j = d\nu_j/d\nu$, $j \geq 1$, and define selection functions

$$P_1^{\text{old}}(\mathbf{s}, s_i) = \sum_{\ell \geq 1} s_i^\ell (1 - s_i) f_\ell(\mathbf{s}), \quad P_{k+1}^{\text{old}}(\mathbf{s}, s_i) = \frac{\sum_{\ell \geq k+1} s_i^\ell (1 - s_i) f_\ell(\mathbf{s})}{\sum_{\ell \geq k} s_i^\ell (1 - s_i) f_\ell(\mathbf{s})},$$

$$P_{k+1}^{\text{new}}(\mathbf{s}, s_i, s_j) = \frac{s_i^k s_j f_k(\mathbf{s})}{\sum_{\ell \geq k} s_i^\ell (1 - s_i) f_\ell(\mathbf{s})}.$$

Procedure 2 (0) To sample Σ_1 in the whole CRT $(\mathcal{T}_{1, \emptyset}, \mu_{1, \emptyset}, \rho_{1, \emptyset}, d_{1, \emptyset}) = (\mathcal{T}, \mu, \rho, d)$ we use step (k, \emptyset) for $k = 1$.

(k, \emptyset) Sample leaf Σ_k in $\mathcal{T}_{k, \emptyset}$ according to Procedure 1 using the SPF P_1^{old} . Then increase k by 1, set $B = [k - 1]$ and $\mathcal{T}_{k, B} = \mathcal{T}$, and proceed to step (k, B) .

(k, B) 1. We are provided with Σ_i , $i \in B \neq \emptyset$, embedded in $\mathcal{T}_{k, B}$, and denote the branch point that separates the labels in B into several subtrees by $v_{k, B}$, given by

$$[[\rho_{k, B}, v_{k, B}]] = \bigcap_{i \in B} [[\rho_{k, B}, \Sigma_i]].$$

2. Conditionally given $(\mathcal{T}_{k, B}; \Sigma_i, i \in B)$, we consider the spine $[[\rho_{k, B}, v_{k, B}]] = \{v_{k, B}^0(t), 0 \leq t < \zeta_{v_{k, B}}\}$ and pick as $\mathcal{T}_{k, B'}$ either a new subtree \mathcal{S} above some $v_{k, B}^0(t)$ with probability

$$\mathbb{P} \left(\mathcal{T}_{k, B'} = \mathcal{S} \mid \mathcal{T}_{k, B}; \Sigma_i, i \in B \right) = P_{\#B}^{\text{new}} \left(S^{v_{k, B}}(t), F_{v_{k, B}}(t), \frac{\mu_{k, B}(\mathcal{S})}{\mu_{k, B}(\mathcal{T}_{(v_{k, B})}^0(t-))} \right) \prod_{t' < t} P_{\#B}^{\text{old}}(S^{v_{k, B}}(t'), F_{v_{k, B}}(t')),$$

or a new or old subtree \mathcal{S} above $v_{k, B}$ with probability

$$\mathbb{P} \left(\mathcal{T}_{k, B'} = \mathcal{S} \mid \mathcal{T}_{k, B}; \Sigma_i, i \in B \right) = \frac{\mu_{k, B}(\mathcal{S})}{\mu_{k, B}(\mathcal{T}_{(v_{k, B})}^0(t-))} \prod_{t' < \zeta_{v_{k, B}}} P_{\#B}^{\text{old}}(S^{v_{k, B}}(t'), F_{v_{k, B}}(t')),$$

where $B' = \{i \in B : \Sigma_i \in \mathcal{S}\}$ and new/old means with/without any Σ_i , $i \in B$.

3. Let $\tau_{k, B} = \inf\{t \geq 0 : \mathcal{T}_{k, B'} \cap \mathcal{T}_{v_{k, B}}^0(t) = \emptyset\}$. We turn $\mathcal{T}_{k, B'}$ into a CRT with rescaled mass measure, root and rescaled distance function as follows:

$$\mu_{k, B'} = \frac{\mu_{k, B} \mid_{\mathcal{T}_{k, B'}}}{\mu_{k, B}(\mathcal{T}_{k, B'})}, \quad \rho_{k, B'} = v_{k, B}^0(\tau_{k, B}), \quad d_{k, B'} = \frac{d_{k, B} \mid_{\mathcal{T}_{k, B'} \times \mathcal{T}_{k, B'}}}{(\mu_{k, B}(\mathcal{T}_{k, B'}))^\alpha}.$$

4. Repeat within the subtree $(\mathcal{T}_{k, B'}, \mu_{k, B'}, \rho_{k, B'}, d_{k, B'})$ by proceeding to step (k, B') .

From Proposition 18, we obtain the following by straightforward arguments.

Corollary 20 *Sample $(\Sigma_k, k \geq 1)$ following Procedure 2. Let v_k be the branch point in \mathcal{T} that separates $[k]$ into different subtrees, $k \geq 1$. Then $\left((S^{\Sigma_k}(t), F_{\Sigma_k}(t), \mathcal{B}_{(\Sigma_k)}^0(t)), 0 \leq t < \zeta_{v_k} \right)$ is a Poisson point process with killing rate $\lambda_k = \int_{S^\downarrow} \sum_{i=1}^{\infty} \sum_{\ell=1}^{k-1} s_i^\ell (1-s_i) \nu_\ell(ds)$ and intensity measure*

$$\tilde{\nu}_{\text{bush}}^{(k)}(ds \otimes dx \otimes dT) = \sum_{i=1}^{\infty} \delta_{s_i}(dx) Q_{(s_1, \dots, s_{i-1}, s_{i+1}, \dots)}(dT) \sum_{\ell=k}^{\infty} s_i^\ell (1-s_i) \nu_\ell(ds). \quad (9)$$

Note that $\lambda_1 = 0$, so in this case, the Poisson point process is not killed and Corollary 20 describes the whole tree \mathcal{T} jointly with Σ_1 , decomposed along its spine $[[\rho, \Sigma_1[[$. For $k \geq 2$, Corollary 20 describes a spinal decomposition along $[[\rho, v_k[[$, except that the subtrees above v_k are not described. We will do this in Lemma 21 below, using more refined arguments.

Proof. The case $k = 1$ follows straight from step $(1, \emptyset)$ of Procedure 2 and Proposition 18. We then proceed by induction in k . Assuming that the statement is true for k , step $(k+1, [k])$ 2. and standard thinning with probabilities $P_{k+1}^{\text{old}}(\mathbf{s}, s_i)$ yields

$$\tilde{\nu}_{\text{bush}}^{(k+1)}(ds \otimes dx \otimes dT) = \sum_{i=1}^{\infty} P_{k+1}^{\text{old}}(\mathbf{s}, s_i) \delta_{s_i}(dx) Q_{(s_1, \dots, s_{i-1}, s_{i+1}, \dots)}(dT) \sum_{\ell=k}^{\infty} s_i^\ell (1-s_i) \nu_\ell(ds),$$

as claimed, and an extra rate $\int_{S^\downarrow} \sum_{i \geq 1} (1 - P_{k+1}^{\text{old}}(\mathbf{s}, s_i)) \sum_{\ell=k}^{\infty} s_i^\ell (1-s_i) \nu_\ell(ds)$ is added to the killing rate λ_k from the induction hypothesis. This completes the induction step. \square

To identify the distribution $Q_1^{[k]}$ of $(\mathcal{T}; \Sigma_i, i \in [k])$ constructed according to Procedure 2 run up to some $k \geq 2$, we study its branching structure recursively by specifying the first branch point v_k that separates $[k]$ into several subtrees denoted by $\mathcal{T}_\ell^{[k]}$ with label partition $\Pi^{[k]}$ and a remaining bush $\mathcal{B}_{[k]}$ of unlabelled subtrees, with joint relative subtree sizes $S^{[k]} \in S^\downarrow$. For $x \in (0, 1]$ and $B = \{b_1, \dots, b_k\} \subset \mathbb{N}$ with $1 \leq b_1 < \dots < b_k$, it will be convenient to denote by Q_x^B the distribution of a rescaled and relabelled version of $(\mathcal{T}; \Sigma_i, i \in [k])$, where the mass measure has been multiplied by x , the distance function by x^α , and Σ_i is renamed to Σ_{b_i} , $i \in [k]$.

Lemma 21 *The first branching of $(\mathcal{T}; \Sigma_i, i \in [k])$ separating $[k]$ and associated subtrees described in $\mathcal{C}_k^{\text{br}} = (S^{[k]}, \Pi^{[k]}, \mathcal{T}^{[k]}, \mathcal{B}_{[k]})$ are independent of $\mathcal{C}_k^{\text{pre}} = ((S^{v_k}(t), F_{v_k}(t), \mathcal{B}_{(v_k)}^0(t)), 0 \leq t < \zeta_{v_k})$, with distribution given by*

$$\begin{aligned} \mathbb{P}(S^{[k]} \in ds, \Pi^{[k]} = \pi, (\mathcal{T}_1^{[k]}; \Sigma_i, i \in \pi_1) \in dT_1, \dots, (\mathcal{T}_r^{[k]}; \Sigma_i, i \in \pi_r) \in dT_r, \mathcal{B}_{[k]} \in dB') \\ = \frac{1}{\lambda_k} \left(\sum_{i_1, \dots, i_r \text{ distinct}} Q_{\widehat{\mathbf{s}}^{(i_1, \dots, i_r)}}(dB') \prod_{\ell=1}^r s_{i_\ell}^{\#\pi_\ell} Q_{s_{i_\ell}}^{\pi_\ell}(dT_\ell) \right) \nu_m(ds), \end{aligned}$$

where $\pi = (\pi_1, \dots, \pi_r) \in \mathcal{P}_k$ and $m = \min \pi_2 - 1$, also $\widehat{\mathbf{s}}^{(i_1, \dots, i_r)}$ is \mathbf{s} with s_1, \dots, s_{i_r} removed.

The kernel $\kappa_{\mathbf{s}, \pi}(dT_1 \otimes \dots \otimes dT_r \otimes dB') = \sum_{i_1, \dots, i_r \text{ distinct}} Q_{\widehat{\mathbf{s}}^{(i_1, \dots, i_r)}}(dB') \prod_{\ell=1}^r s_{i_\ell}^{\#\pi_\ell} Q_{s_{i_\ell}}^{\pi_\ell}(dT_\ell)$ is a fancy paintbox that equips each block under $\kappa_{\mathbf{s}}$ with a tree and embeds the labels for $\pi \in \mathcal{K}$.

Proof. For $k = 1$, this is trivial since $v_1 = \Sigma_1$ is a leaf. Now suppose that the result holds for all $[j] \subseteq [k]$, and consider $k + 1$. In our use of standard Poisson point process arguments as well as in extracting from Procedure 2 as from Procedure 1, we build on the proof of Proposition 18.

For $\pi \in \mathcal{P}_{k+1} \setminus \{\mathbf{1}_{[k+1]}\}$, let $A_\pi = \{\Pi^{[k+1]} = \pi\}$ be the event that v_{k+1} splits $[k+1]$ into π . The simplest case is for $\pi = \{[k], \{k+1\}\}$. By Corollary 20, the decomposition of \mathcal{T} along the

trunk $[[\rho, v_k[[$ is given by the Poisson point process $\left(\left(S^{\Sigma_k}(t), F_{\Sigma_k}(t), \mathcal{B}_{(\Sigma_k)}^0(t) \right), 0 \leq t < \zeta_{v_k} \right)$ with intensity measure (9), killed at rate $\lambda_k = \int_{S^1} \sum_{i=1}^{\infty} \sum_{\ell=1}^{k-1} s_i^\ell (1-s_i) \nu_\ell(ds)$. By comparison with the statement of Corollary 20 for $k+1$, we see $\mathbb{P}(A_{\{[k], \{k+1\}\}}) = 1 - \lambda_k / \lambda_{k+1}$. *Conditionally given* $A_{\{[k], \{k+1\}\}}$, the distribution of $(S^{\Sigma_k}(\tau_{k+1, [k]}), F_{\Sigma_k}(\tau_{k+1, [k]}), \mathcal{B}_{(\Sigma_k)}^{\text{rem}}(\tau_{k+1, [k]}), \mathcal{T}_{k+1, [k]}^{\text{sel}}(\tau_{k+1, [k]}))$ is

$$\begin{aligned} & \frac{1}{\lambda_{k+1} - \lambda_k} \sum_{i \geq 1} \sum_{j \neq i} P_k^{\text{new}}(\mathbf{s}, s_i, s_j) \delta_{s_i}(dx) Q_{\mathfrak{S}^{(i,j)}}(dB') Q_{s_j}(dT'') \sum_{\ell=k}^{\infty} s_i^\ell (1-s_i) \nu_\ell(ds) \\ &= \frac{1}{\lambda_{k+1} - \lambda_k} \sum_{i=1}^{\infty} \sum_{j \neq i} \delta_{s_i}(dx) Q_{\mathfrak{S}^{(i,j)}}(dB') Q_{s_j}(dT'') s_i^k s_j \nu_k(ds), \end{aligned} \quad (10)$$

independently of the rescaled $(\mathcal{T}_{(\Sigma_k)}^0(\tau_{k+1, [k]}); \Sigma_i, i \in [k])$ that has $Q_1^{[k]}$ as conditional distribution given $A_{\{[k], \{k+1\}\}}$. Note also, that the embedding of Σ_{k+1} in the rescaled $\mathcal{T}_{k+1, [k]}^{\text{sel}}(\tau_{k+1, [k]})$ yields conditional distribution $Q_1^{\{k+1\}}$ given $A_{\{[k], \{k+1\}\}}$, and that by standard thinning arguments these are conditionally independent of $\left(\left(S^{\Sigma_{k+1}}(t), F_{\Sigma_{k+1}}(t), \mathcal{B}_{(\Sigma_{k+1})}^0(t) \right), 0 \leq t < \zeta_{v_{k+1}} \right)$ given $A_{\{[k], \{k+1\}\}}$. Multiplying by $\mathbb{P}(A_{\{[k], \{k+1\}\}})$, this yields the result for $\pi = \{[k], \{k+1\}\}$.

Now consider any other $\pi = \{\pi_1, \dots, \pi_r\} \in \mathcal{P}_{k+1} \setminus \{\mathbf{1}_{[k+1]}\}$ and write $m = \min \pi_2 - 1 \in [k-1]$. Note that also $m = \min \pi_2 \cap [k] - 1$. By the induction hypothesis, the collections $\mathcal{C}_k^{\text{pre}}$ describing the spine to the branch point separating $[k]$, and $\mathcal{C}_k^{\text{br}}$ describing the branching and rescaled subtrees, are independent. We read and analyse Step 2. of Procedure 2 by marking $\mathcal{C}_k^{\text{pre}}$ as we marked the Poisson point process in the proof of Proposition 18 *and* similarly and independently selecting a new or old subtree \mathcal{S} above v_k with probability

$$\mathbb{P}\left(\mathcal{T}^{\text{sel}} = \mathcal{S} \mid \mathcal{T}_{k,B}; \Sigma_i, i \in B\right) = \frac{\mu_{k,B}(\mathcal{S})}{\mu_{k,B}\left(\mathcal{T}_{(v_k,B)}^0(t-)\right)}.$$

Then A_π is an intersection of two independent events $A_\pi = A_k^{\text{pre}} \cap A_\pi^{\text{br}}$ given by

$$A_k^{\text{pre}} = \{\mathcal{T}_{v_k}^{\text{sel}} = \{0\} \text{ for all } 0 \leq t < \zeta_{v_k}\} \quad \text{and} \quad A_\pi^{\text{br}} = \{\mathcal{L}_k(\mathcal{T}^{\text{sel}}) = \pi_{(k+1)} \cap [k]\},$$

where $\mathcal{L}_k(\mathcal{S}) = \{i \in [k] : \Sigma_i \in \mathcal{S}\}$ and $\pi_{(k+1)}$ is the block of π containing $k+1$. By construction, $(\mathcal{C}_k^{\text{pre}}, A_k^{\text{pre}})$ and $(\mathcal{C}_k^{\text{br}}, A_\pi^{\text{br}})$ are also independent and, since the random variables used to embed Σ_{k+1} in \mathcal{T}^{sel} are conditionally independent of $(\mathcal{C}_k^{\text{pre}}, A_k^{\text{pre}})$ given \mathcal{T}^{sel} , also $\mathcal{C}_{k+1}^{\text{br}}$ is independent of $(\mathcal{C}_k^{\text{br}}, A_\pi^{\text{br}})$, hence of $\mathcal{C}_{k+1}^{\text{br}}$, since on A_π^{br} , we have $\mathcal{C}_{k+1}^{\text{br}} = \mathcal{C}_k^{\text{br}}$. The distribution of $\mathcal{C}_{k+1}^{\text{br}}$ now follows from the conditional distribution of \mathcal{T}^{sel} given $\mathcal{C}_k^{\text{br}}$, the recursive nature of Procedure 2 and the stability of the procedure under increasing bijections from $[j]$ to other sets $B \subset \mathbb{N}$ with $\#B = j$ that allows us to apply the induction hypothesis to obtain that the embedding of Σ_{k+1} in the rescaled $\mathcal{T}^{\text{sel}} \sim Q_1^B$ yields a tree with rescaled distribution $Q_1^{B \cup \{k+1\}}$, as required. \square

Corollary 22 (Subtree decomposition along a reduced tree) *The discrete tree shapes T_k , $k \geq 1$, of the reduced trees $R(\mathcal{T}; \Sigma_1, \dots, \Sigma_k)$, $k \geq 1$, are labelled Markov branching trees with*

$$\mathbb{P}(\Pi^{[k]} = \pi) = \frac{1}{\lambda_k} \int_{S^1} \kappa_{\mathfrak{s}}(\mathcal{P}^\pi) \nu_m(ds), \quad \text{where } m = \min \pi_2 - 1. \quad (11)$$

Conditionally given T_k , the processes $((S^B(t), F_B(t), \mathcal{B}_B^0(t)), 0 \leq t < \zeta_B)$, $B \in T_k$, where we parametrise $\{v \in \mathcal{T} : \mathcal{L}_k(\mathcal{T}_v) = B\} = \{v_B(t), 0 \leq t < \zeta_B\}$ in homogenous time, are independent with distributions as in Corollary 20, pushed forward under increasing bijections $\#[B] \rightarrow B$.

Conditionally given T_k , in particular $\Pi^B = \pi^B = (\pi_1^B, \dots, \pi_r^B)$ with $m^B = \min \pi_2^B - 1$, the variables $(S^B, F_{\pi_1^B}, \dots, F_{\pi_r^B}, \mathcal{B}_B)$, $B \in T_k$, are independent of everything else with distribution

$$\frac{1}{\lambda_{\#B} \mathbb{P}(\Pi^B = \pi)} \left(\sum_{i_1, \dots, i_r \text{ distinct}} Q_{\mathfrak{s}^{(i_1, \dots, i_r)}}(dB') \prod_{\ell=1}^r s_{i_\ell}^{\#\pi_\ell} \delta_{s_{i_\ell}}(dx_\ell) \right) \nu_{m^B}(ds).$$

The tree $(\mathcal{T}; \Sigma_1, \dots, \Sigma_k)$ with k leaves embedded via Procedure 2 is a measurable function of the random variables $(T_k; ((F_B, S^B, \mathcal{B}_B), ((S^B(t), F_B(t), \mathcal{B}_B^0(t)), 0 \leq t < \zeta_B)), B \in T_k)$ specified.

Proof of Theorem 6. We will show that Procedure 2 provides an embedding for a restricted exchangeable hierarchy as in Corollary 5, provided that $\int_{S^1} (1 - s_1) \nu(ds) < \infty$ and $\nu(s_0 > 0) = c_j = k_j = 0$, $j \geq 1$.

A restricted exchangeable hierarchy is uniquely determined by its restrictions to $[k]$, $k \geq 1$. But the formula for κ in Corollary 5 is identical to (11), hence the hierarchy constructed via Procedure 2 is a restricted exchangeable hierarchy associated with $(\nu_j, j \geq 1)$ embedded in a CRT with characteristic pair (α, ν) , as required. \square

5 Scaling limits, proof of Theorem 7

5.1 Asymptotics of block numbers in Gnedin's constrained partitions

Before we describe Gnedin's framework and provide a slight extension of his asymptotic study, let us establish the renewal theory result that we need for this.

Lemma 23 *Let $N_t = \#\{n \geq 1 : X_1 + \dots + X_n \leq t\}$ be the renewal process associated with independent and identically distributed $X_j > 0$. Then for all $p \in \mathbb{N}$*

$$\limsup_{t \rightarrow \infty} \mathbb{E} \left(\frac{N_t^p}{t^p} \right) < \infty.$$

This is not a deep result, but we have been unable to find it in the literature and hence provide a proof here. This can no doubt be strengthened to a p th moment renewal theorem extending the first moment Elementary Renewal Theorem, but we will not need such a stronger statement.

Proof. The case $p = 1$ follows directly from the well-known Elementary Renewal Theorem for $\mathbb{E}(N_t)$. To prove the general case inductively, we define $q_j(t) = \mathbb{E}(N_t^j)$ and consider the strong induction hypothesis: for all $t \geq 0$,

$$q_j(t) \leq \sum_{k=1}^j a_{jk} (q_1(t))^k \quad \text{for all } 1 \leq j \leq p-1 \text{ and some } a_{jk} \geq 0. \quad (12)$$

This is trivially true for $p = 1$ and $p = 2$. Let F be the distribution function of X_1 and U be the renewal function i.e.

$$U(t) = \begin{cases} 1 + q_1(t) & t \geq 0, \\ 0 & t < 0. \end{cases}$$

To show the induction step, we condition on the first renewal time X_1 and obtain the renewal equation

$$q_p = F + \sum_{j=1}^{p-1} \binom{p}{j} q_j * F + q_p * F$$

where $*$ denotes convolution, i.e. $V * W(t) := \int_0^t V(t-s)dW(s)$, in the sense of Stieltjes integration. Let $F^{*(m)}$ be the distribution function of $T_m = X_1 + \dots + X_m$, $m \geq 1$. Note that

$$\begin{aligned} \mathbb{E}[N_t^p] &= \mathbb{E} \left[\left(\sum_{m \geq 1} \mathbf{1}_{\{T_m \leq t\}} \right)^p \right] = \mathbb{E} \left[\sum_{m_1=1}^{\infty} \cdots \sum_{m_p=1}^{\infty} \mathbf{1}_{\{T_{\max_1 \leq i \leq p} m_i} \leq t\}} \right] \\ &\leq \sum_{m=1}^{\infty} m^p F^{*(m)}(t) = \sum_{n=0}^{\infty} \sum_{j=1}^k (nk+j)^p F^{*(nk+j)}(t) \leq \sum_{n=0}^{\infty} k((n+1)k)^p F^{*(nk)}(t) \\ &\leq \sum_{n=0}^{\infty} k((n+1)k)^p \left(F^{*(k)}(t) \right)^n, \quad \text{for all } k \geq 1, \end{aligned}$$

where the last step used the monotonicity of $G = F^{(k)}$ in a simple estimate of the form

$$(G^{*(r-1)} * G)(t) = \int_0^t G^{*(r-1)}(t-s)dG(s) \leq \int_0^t (G(t-s))^{r-1}dG(s) \leq (G(t))^r, \quad (13)$$

using induction in r , but which is also probabilistically obvious in its interpretation in terms of random variables. Choose k large enough such that $F^{*(k)}(t) < 1$, then we deduce $t \mapsto q_p(t)$ is locally bounded. Therefore the renewal equation has as its unique locally bounded solution

$$q_p = F * U + \sum_{j=1}^{p-1} \binom{p}{j} q_j * F * U,$$

and particularly $q_1 = F * U$. Then using the induction hypothesis and we obtain

$$q_p \leq F * U + \sum_{j=1}^{p-1} \binom{p}{j} \left(\sum_{k=1}^j a_{jk} q_1^k * F * U \right) \leq q_1 + \sum_{k=1}^{p-1} \left(\sum_{j=k}^{p-1} \binom{p}{j} a_{jk} q_1^k * q_1 \right).$$

Applying an argument like (13) to $G = q_1$ and $r = k + 1$, we see that the induction proceeds.

As the Elementary Renewal Theorem guarantees $\mu = \limsup_{t \rightarrow \infty} q_1(t)/t < \infty$, this completes the proof, since now

$$\limsup_{t \rightarrow \infty} \mathbb{E} \left(\frac{N_t^p}{t^p} \right) = \limsup_{t \rightarrow \infty} \frac{q_p(t)}{t^p} \leq \limsup_{t \rightarrow \infty} \sum_{k=1}^p a_{pk} \frac{(q_1(t))^k}{t^p} = a_{pp} \mu^p < \infty. \quad \square$$

Gnedin [15] introduced a *constrained paintbox* based on a strictly decreasing random sequence $(G_k, k \geq 0)$ in $[0, 1]$ with $G_0 = 1$ and $\lim_{k \rightarrow \infty} G_k = 0$. Specifically, he considers a sequence $(I_n, n \geq 1)$ of independent uniform random variables on $[0, 1]$ independent of (G_k) , but then associates a modified sequence $(\bar{T}_n^\psi, n \geq 1)$ that is constrained so that its lower records follow $(G_k, k \geq 1)$ with multiplicities given by a sequence $\psi = (\psi_k, k \geq 1)$:

- set $\bar{T}_1^\psi = \dots = \bar{T}_{\psi_1}^\psi = G_1$, then we have $n = \psi_1$ modified variables, $K_n^\psi = 1$ record has attained its multiplicity according to ψ and the next record has been attained $R_n^\psi = 0$ times;
- given $(\bar{T}_1^\psi, \dots, \bar{T}_n^\psi)$, $K_n^\psi = k \geq 1$ and $R_n^\psi = r \in \{0, \dots, \psi_{k+1} - 1\}$, proceed as follows
 - if $I_{n+1} \in [G_k, 1]$, let $\bar{T}_{n+1}^\psi = I_{n+1}$, $K_{n+1}^\psi = K_n^\psi$ and $R_{n+1}^\psi = R_n^\psi$;
 - if $I_{n+1} \in [0, G_k)$ and $r \leq \psi_{k+1} - 2$, let $\bar{T}_{n+1}^\psi = G_{k+1}$, $K_{n+1}^\psi = K_n^\psi$ and $R_{n+1}^\psi = R_n^\psi + 1$;
 - if $I_{n+1} \in [0, G_k)$ and $r = \psi_{k+1} - 1$, let $\bar{T}_{n+1}^\psi = G_{k+1}$, $K_{n+1}^\psi = K_n^\psi + 1$ and $R_{n+1}^\psi = 0$.

Eventually each G_k will appear ψ_k times as lower record in $(\bar{T}_n^\psi, n \geq 1)$. Let $J_n^\psi = K_n^\psi + \mathbf{1}_{\{R_n^\psi > 0\}}$ be the number of records attained by the n first terms of the sequence. Gnedin obtains the asymptotics of J_n^ψ when $G_k = Y_1 \cdots Y_k$, where for $k \geq 1$ the Y_k , $k \geq 1$, are independent and identically distributed in $(0, 1)$ with finite logarithmic moments $\mathbb{E}[-\log Y_1]$ and $\text{Var}(-\log Y_1)$. Here we drop the requirement of the finite logarithmic moments.

Lemma 24 *Let $G_k = Y_1 \cdots Y_k$, where the Y_k , $k \geq 1$ are independent and identically distributed in $(0, 1)$. If $\psi = (\psi_k, k \geq 1)$ is such that $\psi_k \in \mathbb{N}$, $k \geq 1$ and*

$$\log \left(\sum_{j=1}^k \psi_j \right) = o(k), \quad \text{as } k \rightarrow \infty,$$

then

$$\lim_{n \rightarrow \infty} \frac{J_n^\psi}{\log n} = \frac{1}{\mathbb{E}[-\log Y_1]}$$

in the sense that this limit vanishes when $\mathbb{E}[-\log Y_1] = \infty$. Furthermore, for every $p \geq 1$,

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[\left(\frac{J_n^\psi}{\log n} \right)^p \right] < \infty.$$

Proof. The case $\text{Var}(-\log Y_1) < \infty$, and implicitly also $\mathbb{E}[-\log Y_1] < \infty$, has been shown in the proof of [15, Proposition 8]. We only need to handle the case when $\mathbb{E}[-\log Y_1] = \infty$. Following Gnedin, we define $J'_n = \#\{k \geq 1 : G_k \geq 1/n\} = \#\{k \geq 1 : \sum_{i=1}^k (-\log Y_i) \leq \log n\}$. According to the Renewal Theorem [13, Theorem 4.1, Chapter 3], $J'_n / \log n \rightarrow 0$ a.s. when $\mathbb{E}[-\log Y_1] = \infty$. Let $I_{1,n} < \cdots < I_{n,n}$ be the order statistics of I_1, \dots, I_n . Define ζ_n by $I_{\zeta_n, n} < 1/n < I_{\zeta_n+1, n}$. According to Gnedin's discussion, J'_n and ζ_n are independent, ζ_n is binomial($n, 1/n$) and $J_n^\psi \leq J'_n + \zeta_n$. By Markov's inequality, we have for all $\epsilon > 0$,

$$\mathbb{P}(\zeta_n > \epsilon \log n) = \mathbb{P}\left(e^{2\zeta_n/\epsilon} > n^2\right) \leq \frac{\mathbb{E}[e^{2\zeta_n/\epsilon}]}{n^2} = \frac{1}{n^2} \left(1 + \frac{e^{2/\epsilon} - 1}{n}\right)^n.$$

Thus, we have $\sum_{n=1}^{\infty} \mathbb{P}(\zeta_n > \epsilon \log n) < \infty$. The Borel-Cantelli Lemma now implies that

$$\lim_{n \rightarrow \infty} \zeta_n / \log n = 0 \quad \text{a.s.}$$

This gives us $\limsup_{n \rightarrow \infty} J_n^\psi / \log n = 0$ when $\mathbb{E}[-\log Y_1] = \infty$.

For every $p \geq 1$, note that

$$\mathbb{E} \left[\left(\frac{J_n^\psi}{\log n} \right)^p \right] \leq \mathbb{E} \left[\left(\frac{J'_n + \zeta_n}{\log n} \right)^p \right] \leq 2^{p-1} \left(\mathbb{E} \left[\left(\frac{J'_n}{\log n} \right)^p \right] + \mathbb{E} \left[\left(\frac{\zeta_n}{\log n} \right)^p \right] \right).$$

The first term is bounded due to Lemma 23 and the second term converges to 0 because the moments of ζ_n are bounded. \square

5.2 Special branch points and their asymptotics

We consider the setting of Theorem 7, where for some fixed $m \geq 1$, we have $\nu_j = \nu_m$ for all $j \geq m$. In this setting, the selection probabilities of Section 4.3 for $k \geq m+1$ become

$$P_k^{\text{old}}(\mathbf{s}, s_i) = s_i \quad \text{and} \quad P_k^{\text{new}}(\mathbf{s}, s_i, s_j) = s_j.$$

It is now easy to see that the sampling procedure in (\mathcal{T}, μ) can be simplified in this setting so as to combine for each $k \geq m$ the steps until $\#B' < m$ into a single selection according to μ .

Procedure 3 Use the steps of Procedure 2, but replace for $k \geq m$ steps $(k, [k-1])$ 1.-2. by

1'. We are provided with $(\mathcal{T}; \Sigma_i \in [k])$ and sample $\Sigma_{k+1}^* \sim \mu$.

2'. We consider the spine $[[\rho, \Sigma_{k+1}^*[[= \{\Sigma_{k+1}^{*0}(t), t \geq 0\}$ in homogeneous time and set

$$\mathcal{T}_{k,B'} = \mathcal{T}_{(\Sigma_{k+1}^*)}^0(\tau_k^*), \text{ where } \tau_k^* = \inf\{t \geq 0 : \#\mathcal{L}_k(\mathcal{T}_{(\Sigma_{k+1}^*)}^0(t)) < m\}, B' = \mathcal{L}_k(\mathcal{T}_{(\Sigma_{k+1}^*)}^0(\tau_k^*)),$$

where $\mathcal{L}_k(\mathcal{S}) = \{i \in [k] : \Sigma_i \in \mathcal{S}\}$ is the number of labels in $\mathcal{S} \subseteq \mathcal{T}$.

Theorem 7 describes the convergence of unlabelled trees. In fact, more is true and it will be instructive to study approximations of the spines $[[\rho, \Sigma_j[[$, $j \geq 1$, in $(\mathcal{T}; \Sigma_i, i \in \mathbb{N})$ by discrete spines $\{B \in \mathcal{T}_n : j \in B\}$, $n \geq j \geq 1$. In the proof of Theorem 7 we will need to control these uniformly in $j \geq 1$. In the exchangeable case of [20], these spines can be regarded as independent uniform samples from a strongly sampling consistent regenerative interval partition. In the restricted exchangeable case here, the analogous partitions are no longer regenerative (except for $j = 1$, and for $j = 2$ if $m = 2$) and the sampling is not uniform. However, both features are still present on parts of the spine and we will cut the spines at certain *special branch points*.

Fix $j \geq 1$. A branch point $v \in [[\rho, \Sigma_j[[$ is called *special in $(\mathcal{T}; \Sigma_i, i \in \mathbb{N})$ for $[[\rho, \Sigma_j[[$ if some or all of the m smallest labels $\mathcal{L}(\mathcal{T}^v)$ in the bush \mathcal{T}^v above v are *not* included in the subtree $\mathcal{T}_{(\Sigma_j)}^\alpha(d(\rho, v))$ above v containing Σ_j . Note that a branch point is special if the m smallest labels split or if j splits from the m smallest labels. Therefore, a branch point that is special for $[[\rho, \Sigma_j[[$ and an element of $[[\rho, \Sigma_{j'}[[$ for some $j' < j$ may not be special for $[[\rho, \Sigma_{j'}[[$. For the analogous notion in $(\mathcal{T}; \Sigma_i, i \in [n])$, for $n \geq j$, we write*

$$N_n^{(j)} = \#\{v \in [[\rho, \Sigma_j[[: v \text{ is a special for } [[\rho, \Sigma_j[[\text{ in } (\mathcal{T}; \Sigma_i, i \in [n])\}$$

for the number of special branch points, and $\tau_n^{(j)} = \inf\{t \geq 0 : \#\mathcal{L}_n(\mathcal{T}_{(\Sigma_j)}^0(t)) < m\}$ for the homogeneous time when the label set first has fewer than m elements. The significance of this time is that up to this time, all branch points that are special in $(\mathcal{T}; \Sigma_i, i \in \mathbb{N})$ will also be special in $(\mathcal{T}; \Sigma_i, i \in [n])$, but this fails afterwards. We introduce $V_n^{(j)} = \inf\{t \geq 0 : \Sigma_n \notin \mathcal{T}_{(\Sigma_j)}^0(t)\}$, the homogeneous time when Σ_n leaves the spine $[[\rho, \Sigma_j[[$. Recall $\bar{\Lambda}^*(z) = \tilde{\nu}^*(S^\downarrow \times (0, e^{-z}))$ and also set $\bar{\Lambda}_k(z) = \tilde{\nu}_{\text{bush}}^{(k)}(S^\downarrow \times (0, e^{-z}) \times \mathbb{H})$, $k \geq 1$, in the notation of Lemma 14 and Corollary 20.

Proposition 25 *Let $(\nu_j, j \geq 1)$ and ν be as in Theorem 6 and $(\mathcal{T}; \Sigma_i, i \in \mathbb{N})$ an embedding according to Procedure 2. Suppose furthermore that there is $m \geq 1$ with $\nu_j = \nu_m$, $j \geq m$, and that $\nu_m(s_1 \leq 1 - \epsilon) = \epsilon^{-\alpha} \ell(1/\epsilon)$. Then,*

(i) *for all $j \geq 1$, we have $N_n^{(j)} / (n^\alpha \ell(n)) \xrightarrow[n \rightarrow \infty]{a.s.} 0$;*

(ii) *for every $p \geq 1$, we have $\limsup_{n \rightarrow \infty} \mathbb{E} \left[\left(N_n^{(n)} / \log n \right)^p \right] < \infty$;*

(iii) *for every $p \geq 1$, there exists a constant C_p^{spec} such that for all $1 \leq j \leq n$ and $x > 0$*

$$\mathbb{P} \left(N_n^{(j)} > 2x \max \left\{ \bar{\Lambda}_1(n^{-1}), \bar{\Lambda}^*(n^{-1}) \right\} \right) < \frac{C_p^{\text{spec}}}{x^p n^{\alpha p - 1}}.$$

Proof. (i) Let us consider $N_n^{(1)}$ first. We will study the asymptotics by relating this to the setting of Lemma 24 and will use the notation there once we have identified $(G_k, k \geq 1)$ and ψ . Recall that $X_{(\Sigma_1)}(V_i^{(1)})$, $i \geq 2$, are the residual masses of the subtrees containing Σ_1 when Σ_i

has left the spine $[[\rho, \Sigma_1[[$. Let $Y_k, k \geq 1$, be independent copies of $X_{(\Sigma_1)}(\tau_m^{(1)})$, the residual mass of the subtree containing Σ_1 at the branch point separating $[m]$, and $G_k = Y_1 \cdots Y_k, k \geq 1$.

We introduce the filtration

$$\mathcal{F}_n^{(1)}(t) = \sigma \left((S^{\Sigma_1}(s), F_{\Sigma_1}(s), \mathcal{B}_{(\Sigma_1)}^0(s), \mathcal{L}_n(\mathcal{B}_{(\Sigma_1)}^0)), s \leq t \right), \quad t \geq 0,$$

of the spinal Poisson point process $(S^{\Sigma_1}, F_{\Sigma_1}, \mathcal{B}_{(\Sigma_1)}^0)$ studied in Proposition 18 augmented by label sets of spinal bushes derived from embedded leaves $\Sigma_1, \dots, \Sigma_n$.

Let $H_n^{(1)} = \#\{\tau_i^{(1)}, m \leq i \leq n\}$. Then $H_m^{(1)} = 1$ is the initial state, we will also consider $(\tau_m^{(1)}, X_{(\Sigma_1)}(\tau_m^{(1)}), \#\mathcal{L}_m(\mathcal{T}_{(\Sigma_1)}^0(\tau_m^{(1)})))$. Now let $n \geq m + 1$ and write $\bar{V}_n^{(j)} = \min\{\tau_n^{(j)}, V_n^{(j)}\}$, $n \geq 1$. Conditionally given $\mathcal{F}_{n-1}^{(1)}(\tau_{n-1}^{(1)})$, in particular $(X_{(\Sigma_1)}(\bar{V}_m^{(1)}), \dots, X_{(\Sigma_1)}(\bar{V}_{n-1}^{(1)}))$, $H_{n-1}^{(1)} = k$ and $\#\mathcal{L}_{n-1}(\mathcal{T}_{(\Sigma_1)}^0(\tau_{n-1}^{(1)})) = \ell$, the argument to establish Procedure 3 can be used to simplify Procedure 2 to only combine the steps until $1 \notin B'$ or $\#B' < m$; so sample a leaf $\Sigma_n^* \sim \mu$, define $V_{n,*}^{(1)} = \inf\{t \geq 0 : 1 \notin \mathcal{L}_{n-1}(\mathcal{T}_{(\Sigma_n^*)}^0(t))\}$ and

- if $V_{n,*}^{(1)} \leq \tau_{n-1}^{(1)}$, set $\mathcal{T}_{n-1, B'} = \mathcal{T}_{(\Sigma_1)}^0(V_{n,*}^{(1)})$, note $H_n^{(1)} = k$, $\tau_n^{(1)} = \tau_{n-1}^{(1)}$, $\#\mathcal{L}_n(\mathcal{T}_{(\Sigma_1)}^0(\tau_n^{(1)})) = \ell$;
- if $V_{n,*}^{(1)} > \tau_{n-1}^{(1)}$ and $\ell < m$, set $\mathcal{T}_{n-1, B'} = \mathcal{T}_{(\Sigma_1)}^0(\tau_{n-1}^{(1)})$, note $H_n^{(1)} = k$, $\tau_n^{(1)} = \tau_{n-1}^{(1)}$, $\#\mathcal{L}_n(\mathcal{T}_{(\Sigma_1)}^0(\tau_n^{(1)})) = \ell + 1$;
- if $V_{n,*}^{(1)} > \tau_{n-1}^{(1)}$ and $\ell = m$, then sampling of Σ_n in the rescaled subtree $\mathcal{T}_{(\Sigma_1)}^0(\tau_{n-1}^{(1)})$ is independent of $\mathcal{F}_{n-1}^{(1)}(\tau_{n-1}^{(1)})$ and by the same procedure as Σ_m is sampled in \mathcal{T} , therefore

$$X_{(\Sigma_1)}(\bar{V}_n^{(1)}) \stackrel{d}{=} X_{(\Sigma_1)}(\bar{V}_{n-1}^{(1)})Y_{k+1} \stackrel{d}{=} G_{k+1},$$

note $H_n^{(1)} = k+1$, $\tau_n^{(1)} - \tau_{n-1}^{(1)} \stackrel{d}{=} \tau_m^{(1)}$ independent of $\mathcal{F}_{n-1}^{(1)}(\tau_{n-1}^{(1)})$, and $\#\mathcal{L}_n(\mathcal{T}_{(\Sigma_1)}^0(\tau_n^{(1)})) < m$.

The third case suggests to consider $(\Psi_k, k \geq 1) \stackrel{d}{=} (m - \#\mathcal{L}_{W_{k+1}}(\mathcal{T}_{(\Sigma_1)}^0(\tau_{W_{k+1}}^{(1)})), k \geq 1)$, where $W_k = \inf\{n \geq 1 : H_n^{(1)} = k\}$, independent of $(G_k, k \geq 1)$. As $(G_k, k \geq 1) \stackrel{d}{=} (X_{(\Sigma_1)}(\bar{V}_{W_k}^{(1)}), k \geq 1)$, it is now straightforward to show that the dynamics of $H_n^{(1)}$ and J_{n-m+1}^Ψ are the same, hence there exists a sequence $(I_i, i \geq 1)$ of independent uniform random variables on $[0, 1]$ and an independent random sequence Ψ , each member taking values in $[m]$ such that for all $n \geq m$

$$(H_m^{(1)}, \dots, H_n^{(1)}) \stackrel{d}{=} (J_1^\Psi, \dots, J_{n-m+1}^\Psi). \quad (14)$$

Now note that $n \in \mathbb{N}$ with $W_k < n < W_{k+1}$ can only yield a new special branch point if $V_{n,*}^{(1)} > \tau_{n-1}^{(1)}$, i.e. in the middle case of the procedure above, but after at most $m - 1$ such steps, the third case will apply and $H_n^{(1)}$ will increase. Therefore,

$$N_n^{(1)} \leq m \#H_n^{(1)}. \quad (15)$$

Lemma 24 ensures $H_n^{(1)}/\log n \rightarrow 1/\mathbb{E}[-\log Y_1]$, therefore $N_n^{(1)}/(n^\alpha \ell(n)) \rightarrow 0$ a.s. as $n \rightarrow \infty$.

Now consider $N_n^{(j)}$. For $j \geq 2$, let $v_j = \Sigma_j^0(\lambda_j^0)$ be the spinal branch point at homogeneous time $\lambda_j^0 = \inf\{t \geq 0 : \mathcal{L}_j(\mathcal{T}_{(\Sigma_j)}^0(t)) = \{j\}\}$, when j becomes the smallest label of its block; note that $\Sigma_1, \dots, \Sigma_{j-1}$ are in spinal bushes of $[[\rho, \Sigma_j]]$ below or at v_j . Clearly $\#\mathcal{L}_n(\mathcal{T}_{(\Sigma_j)}^0(\lambda_j)) \leq$

$n - j + 1$, so the number of special branch points on $]]v_j, \Sigma_j]]$ will be no larger than $\tilde{N}_{n-j+1}^{(1)}$ where $(\tilde{N}_k^{(1)}, k \geq 1) \stackrel{d}{=} (N_k^{(1)}, k \geq 1)$. The number of special branch points $[[\rho, v_j]]$ is $N_j^{(j)}$. Therefore

$$N_n^{(j)} \leq N_j^{(j)} + \tilde{N}_{n-j+1}^{(1)} \leq j + \tilde{N}_{n-j+1}^{(1)}. \quad (16)$$

Hence the convergence for $N_n^{(j)}/(n^\alpha \ell(n))$ follows.

(ii) To study $N_n^{(n)}$, we will identify new families $(G_k, k \geq 1)$ and ψ different from the ones in (i) and again apply Lemma 24. Let $b_1^{(j)} = \Sigma_j^0(\chi_1^{(j)})$ be the first special branch point in the spine $[[\rho, \Sigma_j]]$. By Procedure 3, $X_{(\Sigma_j)}(\chi_1^{(j)}) = X_{(\Sigma_j^*)}(\chi_1^{(j)})$ for all $j \geq m + 1$. Also, note that $\chi_1^{(j)}$ is determined by $(\mathcal{T}; \Sigma_i, i \in [m]; \Sigma_j^*)$. As Σ_j^* is sampled according to μ in \mathcal{T} , we have

$$X_{(\Sigma_j)}(\chi_1^{(j)}) = X_{(\Sigma_j^*)}(\chi_1^{(j)}) \stackrel{d}{=} X_{(\Sigma_{m+1}^*)}(\chi_1^{(m+1)}) = X_{(\Sigma_{m+1})}(\chi_1^{(m+1)}). \quad (17)$$

Let $Y_k, k \geq 1$, be independent copies of $X_{\Sigma_{m+1}}(\chi_1^{(m+1)})$ and consider a constrained painbox associated with $G_k = Y_1 \cdots Y_k, k \geq 1$, also $\psi_k = 1, k \geq 1$. We claim that for all $n \geq m + 1$, and every $x > 0$,

$$\mathbb{P}(N_n^{(n)} - m + 1 > x) \leq \mathbb{P}(J_{n-m}^\psi > x). \quad (18)$$

This formula holds for $n = m + 1$ as $N_{m+1}^{(m+1)} - m + 1 \leq J_1^\psi = 1$. Suppose (18) holds for all $n \leq j - 1$. For $n = j$, the first special branch point $b_1^{(j)}$ on the spine $[[\rho, \Sigma_j]]$ is located on the spine $[[\rho, b_1^{(1)}]]$. For $i = m + 1, \dots, j - 1$, let $\mathcal{T}_{(\Sigma_i)}^0(V_i^{(1)} \wedge \chi_1^{(1)})$ be the spinal subtree of \mathcal{T} containing Σ_i rooted on a branch point on the spine $[[\rho, b_1^{(1)}]]$, possibly at $b_1^{(1)}$ itself. By Procedure 3, $\Sigma_i^* \in \mathcal{T}_{(\Sigma_i)}^0(V_i^{(1)} \wedge \chi_1^{(1)})$. We can express the number $M_1^{(j)}$ of leaves in $\{\Sigma_{m+1}, \dots, \Sigma_{j-1}\}$ belonging to the subtree containing Σ_j above branch point $b_1^{(j)}$ as

$$M_1^{(j)} = \#\{i \in \{m+1, \dots, j-1\} : \Sigma_i \in \mathcal{T}_{(\Sigma_j)}^0(\chi_1^{(j)})\} = \#\{i \in \{m+1, \dots, j-1\} : \Sigma_i^* \in \mathcal{T}_{(\Sigma_j)}^0(\chi_1^{(j)})\}.$$

As $\Sigma_{m+1}^*, \dots, \Sigma_{j-1}^*$ are sampled according to μ and $X_{(\Sigma_j)}(\chi_1^{(j)}) \stackrel{d}{=} X_{(\Sigma_{m+1})}(\chi_1^{(m+1)}) \stackrel{d}{=} Y_1$, by (17),

$$\begin{aligned} \mathbb{P}(M_1^{(j)} = k) &= \mathbb{E} \left[\binom{j-m-1}{k} \left(X_{(\Sigma_j)}(\chi_1^{(j)}) \right)^k \left(1 - X_{(\Sigma_j)}(\chi_1^{(j)}) \right)^{j-m-k-1} \right] \\ &= \mathbb{E} \left[\binom{j-m-1}{k} Y_1^k (1 - Y_1)^{j-m-k-1} \right] = \mathbb{P}(\overline{M}_{j-m}^\psi = k) \end{aligned}$$

for all $0 \leq k \leq j - m - 1$, where \overline{M}_{j-m}^ψ is the number of $\overline{I}_1^\psi, \dots, \overline{I}_{j-m}^\psi$ hitting the interval $(0, G_1)$.

Let $N_j^{(j)}(\chi_1^{(j)}, \infty) = N_j^{(j)} - 1$ be the number of special branch points in $]]b_1^{(j)}, \Sigma_j]]$, and $J_{j-m}^\psi(0, Y_1) = J_{j-m}^\psi - 1$. Given $M_1^{(j)} = k$, we have $\#\mathcal{L}_j(\mathcal{T}_{(\Sigma_j)}^0(\chi_1^{(j)})) \leq k + m \leq j - 1$. Hence, we obtain by the induction hypothesis applied to the rescaled $(\mathcal{T}_{(\Sigma_j)}^0(\chi_1^{(j)}); \Sigma_i, i \in \#\mathcal{L}_j(\mathcal{T}_{(\Sigma_j)}^0(\chi_1^{(j)})))$

$$\begin{aligned} &\mathbb{P} \left(N_j^{(j)}(\chi_1^{(j)}, \infty) - m + 1 > x \mid M_1^{(j)} = k, \right) \\ &\leq \mathbb{P} \left(N_{k+m}^{(k+m)} - m + 1 > x \right) \leq \mathbb{P} \left(J_k^\psi > x \right) = \mathbb{P} \left(J_{j-m}^\psi(0, Y_1) > x \mid \overline{M}_{j-m}^\psi = k \right), \end{aligned}$$

and then

$$\begin{aligned} \mathbb{P} \left(N_j^{(j)} - m + 1 > x \right) &= \mathbb{E} \left[\mathbb{P} \left(N_j^{(j)}(\chi_1^{(j)}, \infty) - m + 1 > x - 1 \mid M_1^{(j)} \right) \right] \\ &\leq \mathbb{E} \left[\mathbb{P} \left(J_{j-m}^\psi(0, Y_1) > x - 1 \mid \overline{M}_{j-m}^\psi \right) \right] = \mathbb{P} \left(J_{j-m}^\psi > x \right). \end{aligned}$$

Now (18) is clear and we deduce that $\mathbb{E} \left[(N_n^{(n)} - m + 1)^p \right] \leq \mathbb{E} \left[\left(J_{n-m}^\psi \right)^p \right]$ for every $p \geq 1$. The result in (ii) now follows from Lemma 24.

(iii) Formula (16) implies that for every $p \geq 1$ and $x > 0$ and $z_n = x \max\{\Lambda_1(n^{-1}), \Lambda^*(n^{-1})\}$

$$\begin{aligned} \mathbb{P} \left(N_n^{(j)} > 2z_n \right) &\leq \mathbb{P} \left(N_j^{(j)} > z_n \right) + \mathbb{P} \left(\tilde{N}_{n-j+1}^{(1)} > z_n \right) \\ &\leq \frac{\mathbb{E} \left[(N_j^{(j)})^p \right]}{z_n^p} + \frac{\mathbb{E} \left[(\tilde{N}_{n-j+1}^{(1)})^p \right]}{z_n^p} \leq \frac{C_p (\log n)^p}{z_n^p}. \end{aligned}$$

The last line is obtained by Markov's inequality. Lemma 24 and the result in (ii) gives the upper bound for the first probability, while (15) together with Lemma 24 gives the upper bound of the second one. As $\bar{\Lambda}_1(y) \sim y^{-\alpha} \ell(y)$ and $\bar{\Lambda}^*(y) \sim y^{-\alpha} \ell(y)$ as $y \downarrow 0$, the result in (iii) follows. \square

Procedure 3 and the notion of special branch points are also useful to show that the embedding uses the whole CRT (\mathcal{T}, μ) and does not leave any subtrees of positive mass unlabelled. One way of making this precise is to say that the reduced trees converge to the CRT:

Proposition 26 *In the setting of Procedure 3, we have*

$$R(\mathcal{T}; \Sigma_i, i \in [k]) \rightarrow \mathcal{T} \quad \text{a.s. in the Gromov-Hausdorff sense as } k \rightarrow \infty.$$

Proof. Let $\varepsilon > 0$. Consider $[[\rho, \Sigma_1[[$ and the associated spinal mass partition [21]. Here we denote by $\nu_\varepsilon^{\text{sp}}$ the distribution on S^\downarrow of the masses of spinal subtrees that are greater than ε . Let $\sigma_\varepsilon^{(1)} = \inf\{t \geq 0 : \mu(\mathcal{T}_{(\Sigma_1)}^0(t)) < \varepsilon\}$. Note that $W_1 := \inf\{n \geq 1 : \tau_n^{(1)} \geq \sigma_\varepsilon^{(1)}\} < \infty$ a.s., by the previous proof. By Procedure 3, leaves Σ_n^* and Σ_n are in the same subtree of $[[\rho, \Sigma_1^0(\sigma_\varepsilon)]]$ for each $n > W_1$, in particular each subtree of mass greater than ε is selected with an asymptotic frequency greater than ε . Inductively, we use Corollary 20 and leaves selected according to Procedure 3 to further split according to scaled $\nu_\varepsilon^{\text{sp}}$ each subtree of mass greater than ε .

After a finite number of steps, all subtrees have mass less than ε , e.g. because a homogeneous mass fragmentation process $(F_t, t \geq 0)$ in S^\downarrow with finite dislocation measure $\nu_\varepsilon^{\text{sp}}$ satisfies $F_t \rightarrow 0$ as $t \rightarrow \infty$, see e.g. [7, Equation (4)], and so only has finitely many splits before $|F_1(t)| < \varepsilon$. \square

Using arguments of [30, Corollary 23], we can also show joint a.s. convergence in the Gromov-Prohorov sense of weighted trees $(R(\mathcal{T}; \Sigma_i, i \in [n]), n^{-1} \sum_{i=1}^n \delta_{\Sigma_i}) \rightarrow (\mathcal{T}, \mu)$.

5.3 Convergence of reduced trees and large deviation estimates for spines

By Corollary 22, reduced trees $R(\mathcal{T}; \Sigma_i, i \in [k])$ of self-similar CRTs with labelled leaves embedded according to Procedure 3, can be assigned subtree masses on edges (parts of spines) in terms of Poisson point processes and associated spinal subordinators, and away from existing leaves, sampling of new leaves is according to subtree masses. To study the asymptotics of the number of spinal branchpoints, we will need the following refinement of results in [16, 20].

Lemma 27 *Let $\xi = (\xi_t, t \geq 0)$ be a pure jump subordinator with Lévy measure Λ satisfying $\Lambda([x, \infty)) = x^{-\alpha} \ell(1/x)$, $x \downarrow 0$. Let $(\varepsilon, \tau, \tau')$ be any random variables on $[0, \infty)^2 \times [0, \infty]$ with $\tau \leq \tau'$. Let $(V_i, i \geq 1)$ be any random variables conditionally independent given (ξ, ε, τ) with*

$$\mathbb{P}(V_i \leq \tau | \xi, \varepsilon, \tau) = 1 - e^{-\varepsilon} \quad \text{and} \quad \mathbb{P}(V_i > \tau + v | \xi, \varepsilon, \tau) = e^{-\varepsilon - \xi v}, \quad v \geq 0,$$

and $K_n(\varepsilon, \tau, \tau') = \#\{V_i : 1 \leq i \leq n, \tau < V_i \leq \tau'\}$. Then

$$\lim_{n \rightarrow \infty} \frac{K_n(\varepsilon, \tau, \tau')}{n^\alpha \ell(n) \Gamma(1 - \alpha)} = \int_0^{\tau' - \tau} \exp(-\alpha(\varepsilon + \xi v)) dv \quad \text{a.s. as } n \rightarrow \infty.$$

If furthermore $\Lambda([xy, \infty)) \leq C_\Lambda y^{-\varrho} \Lambda([x, \infty))$ for all $y \geq 1$ and $0 < x \leq 1$, and some $\varrho > 0$, then there is a constant C_p for all $p > 1/\alpha$, such that for all $x \geq 1$, $n \geq 1$ and all $(\varepsilon, \tau, \tau')$ as above, but with the additional property that $\tau' = \tau + \tau''$ for a stopping time τ'' for a filtration, in which ξ is a subordinator,

$$\mathbb{P} \left(\frac{K_n(\varepsilon, \tau, \tau')}{n^{\alpha\ell(n)}\Gamma(1-\alpha)} > (1+x)Y(\varepsilon, \tau, \tau') \right) \leq \frac{C_p}{x^p n^{\alpha p - 1}}, \quad (19)$$

where $Y(\varepsilon, \tau, \tau') = 1 + (1 + A_\alpha)C_\Lambda \sum_{j=0}^{[\tau' - \tau]} \exp(-\varrho(\varepsilon + \xi_j))$ with $A_\alpha = 2 \sum_{j=1}^{\infty} \frac{(j+1)\sqrt{\alpha}}{j(j+1)}$.

This lemma is an extension of [20, Lemmas 8 and 12], which we recover as the special case $\tau = \varepsilon = 0$ and/or $\tau' = \infty$. The proof is also essentially the same, but since this result is more general, we reproduce the proof rewritten in the present generality in the appendix.

Proposition 28 *Let ν_1, \dots, ν_m be conservative with $\nu(s_1 \leq 1 - \epsilon) = \epsilon^{-\alpha} \ell(1/\epsilon)$, where ν is as in Theorem 6 with $\nu_j = \nu_m$, $j \geq m$. Let $R(\mathcal{T}, \Sigma_1, \dots, \Sigma_n)$ be an \mathbb{R} -tree sampled from a self-similar CRT (\mathcal{T}, μ) with index α and dislocation measure ν by Procedure 3, let $(T_n)_{n \geq 1}$ be the associated labelled discrete restricted exchangeable Markov branching trees with unit edge lengths. Then*

$$\frac{R(T_n, [k])}{n^{\alpha\ell(n)}\Gamma(1-\alpha)} \xrightarrow[n \rightarrow \infty]{a.s.} R(\mathcal{T}, \Sigma_1, \dots, \Sigma_k) \quad \text{in the sense that all edge lengths converge.}$$

In particular, the delabelled trees $(R(T_n, [k]))^\circ$, $n \geq k$, converge in the Gromov-Hausdorff sense.

Proof. Consider $k = 1$ and denote by $D_n^{(1)}$ the length of $R(T_n, \{1\})$.

If $\nu_1 = \dots = \nu_{m-1} = 0$, then $\Sigma_1, \dots, \Sigma_m$ are always in the same subtree in \mathcal{T} , then $\tau_1^{(1)} = \dots = \tau_m^{(1)} = \infty$. Conditionally on the subordinator ξ^{Σ_1} associated with leaf Σ_1 , the leaves $\Sigma_{m+1}, \dots, \Sigma_n$ are sampled according to μ along the spine $[[\rho, \Sigma_1[$. Hence applying Lemma 27, the convergence result is straightforward.

Now suppose that at least one of ν_1, \dots, ν_{m-1} is non-zero. By Procedure 3, each Σ_i is either placed in the same subtree of $[[\rho, \Sigma_1[$ as $\Sigma_i^* \sim \mu$ or contributes a special branch point. Now

$$D_n^{(1)} = \# \left\{ V_i^{(1)}, 1 \leq i \leq n \right\} \leq 1 + N_n^{(1)} + \# \left\{ V_{i,*}^{(1)}, 2 \leq i \leq n \right\}, \quad (20)$$

with $V_{i,*}^{(1)} = \inf \{ t \geq 0 : 1 \notin \mathcal{L}_{n-1}(\mathcal{T}_{(\Sigma_n^*)}^0(t)) \}$, where Lemma 27 yields the asymptotics of $K_{n-1}^{(1)}(0, 0, \infty) = \# \{ V_{i,*}^{(1)}, 2 \leq i \leq n \}$. Together with the asymptotics of $N_n^{(1)}$ obtained in Proposition 25, this yields

$$\limsup_{n \rightarrow \infty} \frac{D_n^{(1)}}{n^{\alpha\ell(n)}\Gamma(1-\alpha)} \leq \int_0^\infty \exp(-\alpha \xi_t^{\Sigma_1}) dt \quad \text{a.s.} \quad (21)$$

On the other hand, no special branch points are created for $n \geq l+1 \geq m+2$ below $\tau_l^{(1)}$, so

$$D_n^{(1)} \geq \# \left\{ V_i^{(1)} : 0 < V_i^{(1)} \leq \tau_l^{(1)}, l+1 \leq i \leq n \right\} = \# \left\{ V_{i,*}^{(1)} : 0 < V_{i,*}^{(1)} \leq \tau_l^{(1)}, l+1 \leq i \leq n \right\}.$$

At least one of $\nu_j \neq 0$, $j \leq m-1$, so $\tau_m^{(1)} < \infty$. By the proof of Proposition 25, $\tau_l^{(1)} \rightarrow \infty$, so

$$\liminf_{n \rightarrow \infty} \frac{D_n^{(1)}}{n^{\alpha\ell(n)}\Gamma(1-\alpha)} \geq \sup_{l \geq m+1} \liminf_{n \rightarrow \infty} \frac{\# \{ V_{i,*}^{(1)} : V_{i,*}^{(1)} \leq \tau_l^{(1)}, l+1 \leq i \leq n \}}{n^{\alpha\ell(n)}\Gamma(1-\alpha)} = \int_0^\infty \exp(-\alpha \xi_t^{\Sigma_1}) dt.$$

Combining this with (21), the convergence for $D_n^{(1)}$ follows and establishes the result for $k = 1$.

Next, consider $k \geq 2$ assuming the result for $1, \dots, k-1$. For the branch point v_k adjacent to ρ in $R(\mathcal{T}, \Sigma_1, \dots, \Sigma_k)$, set $D^{[k]} = d(\rho, v_k)$, with homogeneous time ζ_{v_k} given by

$$D^{[k]} = \int_0^{\zeta_{v_k}} \exp(-\alpha \xi_t^{\Sigma_1}) dt.$$

Let $D_n^{[k]}$ be the height of the branch point adjacent to the root in $R(T_n, [k])$, then $D_n^{[k]} - 1$ is the number of distinct branch points of $R(\mathcal{T}, \Sigma_1, \dots, \Sigma_n)$ belonging to $[[\rho, v_k[[$, i.e.

$$D_n^{[k]} = 1 + \#\{V_i^{(1)} : 0 < V_i^{(1)} < \zeta_{v_k}, k+1 \leq i \leq n\}.$$

If $2 \leq k \leq m$, then $1 \leq D_m^{[k]} \leq m-1$ and, by the same argument as for $k=1$,

$$K_{n-m}^{(k)}(0, 0, \zeta_{v_k}) = \#\{V_{i,*}^{(1)} : V_{i,*}^{(1)} < \zeta_{v_k}, m+1 \leq i \leq n\} \leq D_n^{[k]} \leq m + K_{n-m}^{(k)}(0, 0, \zeta_{v_k}). \quad (22)$$

If $k \geq m+1$, then $D_n^{[k]} = 1 + \#\{V_{i,*}^{(1)} : V_{i,*}^{(1)} < \zeta_{v_k}, k+1 \leq i \leq n\}$. In all cases, by Lemma 27

$$\frac{D_n^{[k]}}{n^\alpha \ell(n) \Gamma(1-\alpha)} \xrightarrow[n \rightarrow \infty]{a.s.} \int_0^{\zeta_{v_k}} \exp(-\alpha \xi_s^{\Sigma_1}) ds = D^{[k]}.$$

So the renormalized length of the root edge of $R(T_n, [k])$ converges as required.

Now argue conditionally given that $[k]$ is first separated into $\Pi^{[k]} = (\pi_1, \dots, \pi_r)$. For all $n \geq k+1$ and $1 \leq j \leq r$, denote by $B_j(n) = \mathcal{L}_n(\mathcal{T}_j^{[k]}) \supset \pi_j$ the j th block of the partition at v_k in $(\mathcal{T}; \Sigma_i, i \in [n])$, and by $T_{n,j}^{[k]}$ the corresponding subtree of T_n . By Lemma 21, Procedure 3 and the Strong Law of Large Numbers,

$$\frac{\#B_j(n)}{n} \xrightarrow[n \rightarrow \infty]{a.s.} \mu(\mathcal{T}_j^{[k]}), \quad 1 \leq j \leq r,$$

and the Induction Hypothesis yields convergence of the remaining edge lengths, for $1 \leq j \leq r$

$$\begin{aligned} \frac{R(T_{n,j}^{[k]}, \pi_j)}{n^\alpha \ell(n) \Gamma(1-\alpha)} &= \frac{(\#B_j(n))^\alpha \ell(\#B_j(n))}{n^\alpha \ell(n)} \frac{R(T_{n,j}^{[k]}, \pi_j)}{(\#B_j(n))^\alpha \ell(\#B_j(n)) \Gamma(1-\alpha)} \\ &\xrightarrow[n \rightarrow \infty]{a.s.} (\mu(\mathcal{T}_j^{[k]}))^\alpha R(\mathcal{T}_j^{[k]}, \Sigma_i, i \in \pi_j), \end{aligned}$$

in the sense that all edge lengths converge, which implies Gromov-Hausdorff convergence. \square

While the arguments of the analogous but much more specific [30, Proposition 22] do not apply here in cases where the densities $f_k = d\nu_k/d\nu$ are degenerate, we can now deduce from our Proposition 26 that in the setting of Proposition 28 here, delabelled trees converge a.s. when taking double limits

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{(R(T_n, [k]))^\circ}{n^\alpha \ell(n) \Gamma(1-\alpha)} = \mathcal{T} \quad \text{in the Gromov-Hausdorff sense a.s.} \quad (23)$$

Theorem 7, instead of restricting to $[k]$, then letting $n \rightarrow \infty$ and then $k \rightarrow \infty$, considers $n \rightarrow \infty$ directly, at the cost of weakening the mode of convergence to convergence in probability. To prepare the proof of this otherwise stronger Theorem 7, we study the spines $[[\rho, \Sigma_j[[$, $j \geq 1$.

Recall that we denote by Λ_1 and Λ^* the Lévy measures of the subordinators ξ^{Σ_1} and ξ^{Σ^*} generated, respectively, by the first embedded leaf Σ_1 and by a leaf Σ^* sampled according to μ . For $k \geq 1$ and $n \geq k$, denote by $D_n^{(k)}$ the length of $R(T_n, \{k\})$.

Lemma 29 For all $p \geq 0$, there is a constant $C'_p > 0$ such that for all $k \geq 1$, $n \geq k$ and $x \geq 1$

$$\mathbb{P}\left(D_n^{(k)} > 2(1+x)(2+Z_k) \max\{\bar{\Lambda}_1(n^{-1}), \bar{\Lambda}^*(n^{-1})\}\right) \leq \frac{C'_p}{x^p n^{\alpha p - 1}},$$

where $Z_k = m + (1 + A_\alpha) \max\{C_{\Lambda_1}, C_{\Lambda^*}\} \left(m + \sum_{i=0}^{\infty} (X_{(\Sigma_k)}(i))^\ell\right)$ has all moments finite.

Proof. For $k = 1$, we use (20) to write $D_n^{(1)} \leq 2(D_n^{(1)} - 1) \leq 2N_n^{(1)} + 2K_{n-1}^{(1)}(0, 0, \infty)$ and deduce from Proposition 25 and Lemma 27 that for all $p \geq 0$ and all $n \geq 1$, $x \geq 1$,

$$\begin{aligned} & \mathbb{P}\left(D_n^{(1)} > 2(1+x)(2+Z_1)\bar{\Lambda}_1(n^{-1})\right) \\ & \leq \mathbb{P}\left(N_n^{(1)} > (1+x)2\bar{\Lambda}_1(n^{-1})\right) + \mathbb{P}\left(K_{n-1}^{(1)}(0, 0, \infty) > (1+x)Z_1\bar{\Lambda}_1(n^{-1})\right) \leq \frac{C_p^{\text{spec}} + C_p^{(1)}}{x^p n^{\alpha p - 1}}. \end{aligned}$$

Next, consider $2 \leq k \leq m$. Recall that we denote by $\mathcal{L}_k(\mathcal{S}) = \{i \in [k] : \Sigma_i \in \mathcal{S}\}$ the set of labels in a subtree $\mathcal{S} \subseteq \mathcal{T}$. We set $\gamma_k^{(k)} = 0$ and split the spine $[[\rho, \Sigma_k[[$ at homogeneous times $\gamma_j^{(k)} = \inf\{t \geq 0 : \#\mathcal{L}_k(\mathcal{T}_{(\Sigma_k)}^0(t)) \leq j\}$ for $k-1 \geq j \geq 1$, some of which may coincide. Repeated application of Corollary 20, Lemma 21 and arguing as for (20) yields that

$$D_n^{(k)} \leq 2(D_n^{(k)} - 1) \leq 2N_n^{(k)} + 2K_{n-k}^{(k,1)}\left(\xi^{\Sigma_k}(\gamma_1^{(k)}), \gamma_1^{(k)}, \infty\right) + \sum_{j=2}^k 2K_{n-k}^{(k,j)}\left(\xi^{\Sigma_k}(\gamma_j^{(k)}), \gamma_j^{(k)}, \gamma_{j-1}^{(k)}\right),$$

where $K_{n-k}^{(k,j)}\left(\xi^{\Sigma_k}(\gamma_j^{(k)}), \gamma_j^{(k)}, \gamma_{j-1}^{(k)}\right)$ is as in Lemma 27, but here associated with the subordinator $\xi^{(k,j)} = \xi^{\Sigma_{k,j}}(\gamma_j^{(k)} + \cdot) - \xi^{\Sigma_{k,j}}(\gamma_j^{(k)})$ that has Lévy measure $\Lambda^{(k,j)} = \Lambda_1$ and with random variables $V_i^{(k,j)} = \inf\{t \geq 0 : \Sigma_{k+i}^* \notin \mathcal{T}_{(\Sigma_{k,j})}^0(t)\}$, $i \geq 1$, where $\Sigma_{k,j} = \Sigma_\ell$ if $\ell = \min \mathcal{L}_k(\mathcal{T}_{(\Sigma_k)}^0(\gamma_j^{(k)}))$.

Distribute most of Z_k onto $Z^{(k)}(\gamma_j^{(k)}, \gamma_{j-1}^{(k)}) = 1 + (1 + A_\alpha)C_{\Lambda_1} \sum_{i=[\gamma_{j-1}^{(k)}]^{\lceil \gamma_{j-1}^{(k)} \rceil}}^{\lceil \gamma_{j-1}^{(k)} \rceil} \exp(-\varrho \xi_i^{\Sigma_k})$. Then

$$\begin{aligned} \mathbb{P}\left(D_n^{(k)} > 2(1+x)(2+Z_k)\bar{\Lambda}_1(n^{-1})\right) & \leq \mathbb{P}\left(N_n^{(k)} > 2(1+x)\bar{\Lambda}_1(n^{-1})\right) \\ & + \sum_{j=1}^k \mathbb{P}\left(K_{n-k}^{(k,j)}(\xi^{\Sigma_k}(\gamma_j^{(k)}), \gamma_j^{(k)}, \gamma_{j-1}^{(k)}) > (1+x)Z^{(k)}(\gamma_j^{(k)}, \gamma_{j-1}^{(k)})\Lambda_1(n^{-1})\right) \end{aligned}$$

and conclude again by Proposition 25 and Lemma 27 with constant $C_p^{\text{spec}} + kC_p^{(1)}$.

Now consider $k \geq m+1$. We set $\gamma_{m+1}^{(k)} = 0$ and $\gamma_0^{(k)} = \infty$. We split $[[\rho, \Sigma_k[[$ at homogeneous times $\gamma_m^{(k)} = \inf\{t \geq 0 : \Sigma_k^* \notin \mathcal{T}_{(\Sigma_k)}^0(t)\}$ and $\gamma_j^{(k)} = \inf\{t \geq \gamma_m^{(k)} : \#\mathcal{L}_k(\mathcal{T}_{(\Sigma_k)}^0(t)) \leq j\}$ for $m-1 \geq j \geq 1$. Note that, by Procedure 3, $\#\mathcal{L}_k(\mathcal{T}_{(\Sigma_k)}^0(\gamma_m^{(k)})) \leq m$. Again

$$D_n^{(k)} \leq 2(D_n^{(k)} - 1) \leq 2N_n^{(k)} + \sum_{j=1}^k 2K_{n-m}^{(k,j)}\left(\varepsilon_j^{(k)}, \gamma_j^{(k)}, \gamma_{j-1}^{(k)}\right) + 2K_{n-m}^{(k,*)}(0, 0, \gamma_m^{(k)}),$$

where $\varepsilon_j^{(k)} = \xi^{\Sigma_k}(\gamma_j^{(k)})$, other notation as for $k \leq m$, and $K_{n-m}^{(k,*)}(0, 0, \gamma_m^{(k)})$ is as in Lemma 27, here based on the subordinator $\xi^{\Sigma_k^*}$ with Lévy measure Λ^* , and $V_i^{(k,*)} = \inf\{t \geq 0 : \Sigma_i^* \notin \mathcal{T}_{(\Sigma_k^*)}^0(t)\}$, the homogeneous time when Σ_k^* and Σ_i^* are first in different subtrees. We get

$$\begin{aligned}
\mathbb{P}\left(D_n^{(k)} > 2(1+x)(2+Z_k) \max\{\bar{\Lambda}_1(n^{-1}), \bar{\Lambda}^*(n^{-1})\}\right) &\leq \mathbb{P}\left(N_n^{(k)} > 2(1+x)\bar{\Lambda}_1(n^{-1})\right) \\
&+ \sum_{j=1}^k \mathbb{P}\left(K_{n-m}^{(k,j)}(\xi^{\Sigma_k}(\gamma_j^{(k)}), \gamma_j^{(k)}, \gamma_{j-1}^{(k)}) > (1+x)Z^{(k)}(\gamma_j^{(k)}, \gamma_{j-1}^{(k)})\Lambda_1(n^{-1})\right) \\
&+ \mathbb{P}\left(K_{n-m}^{(k,*)}(0, 0, \gamma_m^{(k)}) > (1+x)Z^{(k)}(0, \gamma_m^{(k)})\bar{\Lambda}^*(n^{-1})\right)
\end{aligned}$$

and conclude again by Proposition 25 and Lemma 27 with constant $C'_p = C_p^{\text{spec}} + mC_p^{(1)} + C_p^*$. Let H_T^ϱ be the height of the ϱ -self-similar CRT $(\mathcal{T}^\varrho, \mu^\varrho)$ obtained from (\mathcal{T}, μ) by ϱ -self-similar time-change. By [18, Proposition 14], the height H_T^ϱ has exponential moments and so does Z_k :

$$\begin{aligned}
\sup_{k \geq 1} Z_k &\leq m + (1 + A_\alpha) \max\{C_{\Lambda_1}, C_{\Lambda^*}\} \left(m + \sup_{k \geq 1} \int_0^\infty (X_{(\Sigma_k)}(t))^\varrho dt \right) \\
&\leq m + (1 + A_\alpha) \max\{C_{\Lambda_1}, C_{\Lambda^*}\} (m + H_T^\varrho). \quad \square
\end{aligned}$$

5.4 Proof of Theorem 7

The previous sections contain the new developments that we need to apply the techniques developed in [20] for the exchangeable case in the higher generality of Theorem 7. We only briefly retrace this argument here so as to identify the places where a result in the previous sections here replaces a more specific result of [20].

Lemma 30 (Lemma 10 and Corollary 11 of [20]) *Let $H_n = \max_{1 \leq k \leq n} D_n^{(k)}$ be the height of T_n . Then there is a constant $C_{p,a}$ for all $a > 0$, $p \geq 2/\alpha$, such that for all $x \geq 1$ and $n \geq 1$*

$$\mathbb{P}\left(\frac{H_n}{\max\{\bar{\Lambda}_1(n^{-1}), \bar{\Lambda}^*(n^{-1})\}} > ax\right) \leq \frac{C_{p,a}}{x^p}.$$

The proof is based on Lemma 29 replacing [20, Lemma 12].

Lemma 31 (Proposition 9 of [20]) *Under the hypotheses of Theorem 7, let for $n \geq k$*

$$\Delta(n, k) := \max_{1 \leq i \leq n} d_n(\{i\}, R(T_n, [k])),$$

d_n being the metric associated with T_n . Then for each $\eta > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\frac{\Delta(n, k)}{\max\{\bar{\Lambda}_1(n^{-1}), \bar{\Lambda}^*(n^{-1})\}} > \eta\right) = 0.$$

The proof is based on Proposition 26 or (23) replacing “clearly”, Corollary 22 replacing [20, reference [10] there, Lemma 3.14], and Lemma 30 replacing [20, Corollary 11].

Proof of Theorem 7. This proof is now based on (23) replacing [20, reference [29]], Lemma 31 replacing [20, Proposition 9], Proposition 28 replacing [20, Proposition 7]. \square

6 Examples: skewed Poisson-Dirichlet models

6.1 The alpha-gamma model as restricted exchangeable hierarchies

The alpha-gamma model [11] is a consistent family $(T_n, n \geq 1)$ of random hierarchies of $[n]$, $n \geq 2$, whose distributions $(Q_n, n \geq 2)$ are such that the conditional distributions of T_{n+1} given T_n are particularly simple. To describe these, recall (see e.g. [26]) that consistency $\mathbf{t}_n = \mathbf{t}_{n+1} \cap [n]$ for two hierarchies \mathbf{t}_n of $[n]$ and \mathbf{t}_{n+1} of $[n+1]$ means that there is a unique $B \in \mathbf{t}_n$ such that either B splits into two vertices B and $B \cup \{n+1\}$, $\{n+1\}$ is added and some vertices renamed

$$\mathbf{t}_{n+1} = \mathbf{t}_n^{B\text{-edge}} = \{A \cup \{n+1\} : A \in \mathbf{t}_n, B \subseteq A\} \cup \{A : A \in \mathbf{t}_n : B \not\subseteq A\} \cup \{B, \{n+1\}\}$$

and we then say that $n+1$ is inserted into the edge below vertex $B \in \mathbf{t}_n$, or

$$\mathbf{t}_{n+1} = \mathbf{t}_n^{B\text{-vertex}} = \{A \cup \{n+1\} : A \in \mathbf{t}_n, B \subseteq A\} \cup \{A : A \in \mathbf{t}_n : B \not\subseteq A\} \cup \{\{n+1\}\}$$

and we then say that $n+1$ is inserted into the internal vertex $B \in \mathbf{t}_n \setminus \mathbf{0}_{[n]}$.

For parameters $0 \leq \alpha \leq 1$ and $0 \leq \gamma \leq \alpha$, we specify $T_1 = \{\{1\}\}$, $T_2 = \mathbf{0}_{[2]} \cup \mathbf{1}_{[2]}$ and for $n \geq 2$

$$\begin{aligned} \mathbb{P}(T_{n+1} = \mathbf{t}_n^{B\text{-edge}} | T_n = \mathbf{t}_n) &= \frac{1-\alpha}{n-\alpha} && \text{for all } B \in \mathbf{t}_n \text{ with } \#B = 1 \\ \mathbb{P}(T_{n+1} = \mathbf{t}_n^{B\text{-edge}} | T_n = \mathbf{t}_n) &= \frac{\gamma}{n-\alpha} && \text{for all } B \in \mathbf{t}_n \text{ with } \#B \geq 2 \\ \mathbb{P}(T_{n+1} = \mathbf{t}_n^{B\text{-vertex}} | T_n = \mathbf{t}_n) &= \frac{(k_n^B - 1)\alpha - \gamma}{n-\alpha} && \text{for all } B \in \mathbf{t}_n \text{ with } \#B \geq 2, \end{aligned}$$

where $k_n^B + 1$ is the degree of vertex $B \in \mathbf{t}_n$, or equivalently $k = k_n^B$ the number of blocks of the partition of B into maximal subsets A_1, \dots, A_k of B in \mathbf{t}_n . Let $Q_n^{\alpha, \gamma}$ be the distribution of T_n .

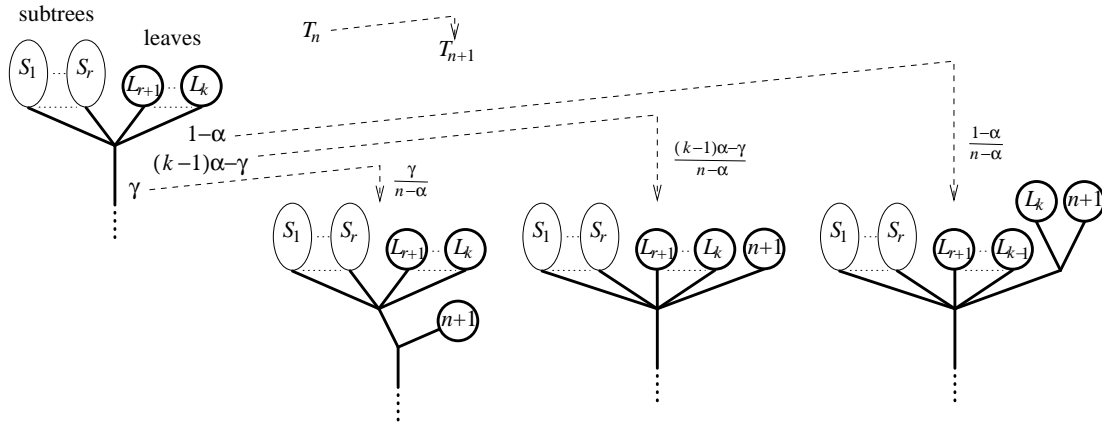


Figure 1: Alpha-gamma growth rule: displayed is one vertex B of T_n with degree $k+1$, hence vertex weight $(k-1)\alpha - \gamma$, with $k-r$ leaves $L_{r+1}, \dots, L_k \in [n]$ and r bigger subtrees S_1, \dots, S_r ; all edges also carry weights, weight $1-\alpha$ and γ are displayed here for the *leaf edge* below $\{L_k\}$ and the *inner edge* below B only; the three associated possibilities for T_{n+1} are displayed.

Proposition 32 *The alpha-gamma model for $\alpha \in [0, 1]$ and $\gamma \in [0, \alpha]$ is a restricted exchangeable Markov branching model with dislocation measures of the form identified in Corollary 5 with $\nu_1 = (1-\alpha)\text{PD}_{\alpha, -\alpha-\gamma}^*$ and $\nu_j = \gamma\text{PD}_{\alpha, -\alpha-\gamma}^*$, $j \geq 2$.*

Here, the Poisson-Dirichlet measure $\text{PD}_{\alpha, \theta}^*(ds)$ as σ -finite measure on S^1 was given by [27, 21]

$$\mathbb{E}[\sigma_1^\theta; \sigma_1^{-1} \Delta \sigma_{[0,1]} \in ds], \quad \theta > -2\alpha, \alpha \in (0, 1),$$

on the interior of the parameter range, where $(\sigma_t, t \geq 0)$ is a stable subordinator with Laplace transform $\mathbb{E}[e^{-\lambda\sigma_t}] = e^{-t\lambda^\alpha}$ and $\Delta\sigma_{[0,1]}$ is the decreasing rearrangements of the jumps $\Delta\sigma_t = \sigma_t - \sigma_{t-}$, $t \in [0, 1]$. A separate, but compatible definition on the boundary of the parameter $\alpha \in \{0, 1\}$ can be given. More importantly, the binary case $\text{PD}_{\alpha, -2\alpha}^*(d\mathbf{s})$ is the ranked beta measure on $\{(x, 1-x, 0, \dots), x \in (1/2, 1)\} \subset S^\downarrow$ with density $2x^{-\alpha}(1-x)^{-\alpha}1_{(1/2, 1)}(x)$; the associated Markov branching model is Aldous's [4] beta-splitting model.

Proof. We claim that the distribution of the partition Π_n of $T_n \sim Q_n^{\alpha, \gamma}$ at $[n]$ is given by

$$\mathbb{P}(\Pi_n = \pi) = \begin{cases} p_n^1(n_1, \dots, n_k) = (1-\alpha) \frac{\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \frac{\alpha^{k-2} \Gamma(k-1-\gamma/\alpha)}{\Gamma(1-\gamma/\alpha)} \prod_{i=1}^k \frac{\Gamma(n_i-\alpha)}{\Gamma(1-\alpha)}, & \pi \in \mathcal{P}_n^1 = \mathcal{P}_n^{\mathbf{0}[2]} \\ p_n^2(n_1, \dots, n_k) = \gamma \frac{\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \frac{\alpha^{k-2} \Gamma(k-1-\gamma/\alpha)}{\Gamma(1-\gamma/\alpha)} \prod_{i=1}^k \frac{\Gamma(n_i-\alpha)}{\Gamma(1-\alpha)}, & \pi \in \mathcal{P}_n^2 = \mathcal{P}_n^{\mathbf{1}[2]} \end{cases}$$

and that $(Q_n^{\alpha, \gamma}, n \geq 2)$ has the labelled Markov branching property

$$\mathbb{P}(\Pi_n = \pi, S_1^n = \mathbf{s}_1, \dots, S_k^n = \mathbf{s}_k) = p_n^j(\#\pi_1, \dots, \#\pi_k) \prod_{i=1}^k Q_{\pi_i}^{\alpha, \gamma}(\{\mathbf{s}_i\}), \quad \pi \in \mathcal{P}_n^j,$$

where $Q_{\pi_i}^{\alpha, \gamma}$ is the push-forward of $Q_{\#\pi_i}^{\alpha, \gamma}$ under the natural bijection on the set of hierarchies induced by the increasing bijection from $[\#\pi_i]$ to π_i .

We show this by induction on n . Specifically, for $n = 2$, this is trivial, for $n = 3$ we have e.g.

$$\mathbb{P}(\Pi_3 = \{\{1, 3\}, \{2\}\}) = \mathbb{P}(\Pi_3 = \{\{1\}, \{2, 3\}\}) = \frac{1-\alpha}{2-\alpha}, \quad \mathbb{P}(\Pi_3 = \{\{1, 2\}, \{3\}\}) = \frac{\gamma}{2-\alpha}.$$

If the claim holds for n , we can apply the growth rules and the induction hypothesis to see

$$\mathbb{P}(\Pi_{n+1} = \{[n], \{n+1\}\}, S_1^{n+1} = \mathbf{s}_1, S_2^{n+1} = \{\{n+1\}\}) = \frac{\gamma}{n-\alpha} Q_n(\{\mathbf{s}_1\}) Q_{\{n+1\}}(\{n+1\}),$$

and for $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}_n^j$, $j = 1, 2$, and hierarchies \mathbf{s}_i of π_i , $i \neq i'$, and $\mathbf{s}_{i'}$ of $\pi_{i'} \cup \{n+1\}$,

$$\begin{aligned} & \mathbb{P}(\Pi_{n+1} = (\pi_1, \dots, \pi_k, \{n+1\}), S_1^{n+1} = \mathbf{s}_1, \dots, S_k^{n+1} = \mathbf{s}_k, S_{k+1}^{n+1} = \{\{n+1\}\}) \\ &= \frac{(k-1)\alpha - \gamma}{n-\alpha} p_n^j(\#\pi_1, \dots, \#\pi_k) Q_{\{n+1\}}(\{\{n+1\}\}) \prod_{i=1}^k Q_{\pi_i}(\{\mathbf{s}_i\}), \\ & \mathbb{P}(\Pi_{n+1} = (\pi_1, \dots, \pi_{i'} \cup \{n+1\}, \dots, \pi_k), S_1^{n+1} = \mathbf{s}_1, \dots, S_k^{n+1} = \mathbf{s}_k) \\ &= \frac{n_{i'} - \alpha}{n-\alpha} p_n^j(\#\pi_1, \dots, \#\pi_k) Q_{\pi_{i'} \cup \{n+1\}}(\{\mathbf{s}_{i'}\}) \prod_{i \neq i'} Q_{\pi_i}(\{\mathbf{s}_i\}), \end{aligned}$$

as conditionally given that the insertion of $n+1$ is in subtree $S_{i'}^n$, it is just as an insertion of $\#\pi_{i'} + 1$ into $T_{\#\pi_{i'}}$, pushed forward from $[\#\pi_{i'} + 1]$ to $\pi_{i'} \cup \{n+1\}$. The induction proceeds. \square

6.2 The skewed Poisson-Dirichlet model extending the alpha-gamma model

Proposition 32 suggests to introduce a three-parameter family of restricted exchangeable fragmentation trees that we call the *skewed Poisson-Dirichlet model*, by setting

$$\nu_1 = \lambda \text{PD}_{\alpha, \theta}^*, \quad \nu_j = (1-\lambda) \text{PD}_{\alpha, \theta}^*, \quad j \geq 2,$$

for $\alpha \in [0, 1]$, $\theta \geq -2\alpha$ and $\lambda \in [0, 1]$. When $\lambda = (1-\alpha)/(1-\theta-2\alpha)$ and $\theta = -\alpha - \gamma$, this is the alpha-gamma model; when $\lambda = 1/2$, this is the exchangeable Poisson-Dirichlet model studied in [26, 21]. We will use parameterisations by $(\alpha, \theta, \lambda)$ and $(\alpha, \gamma, \lambda)$, where $\gamma = -\alpha - \theta$. We can apply Theorem 7 to obtain a strong convergence result:

Corollary 33 *Let $(T_n, n \geq 1)$ be a consistent family of skewed Poisson-Dirichlet trees for parameters $0 < \alpha < 1$, $0 < \gamma = -\alpha - \theta \leq \alpha$ and $0 \leq \lambda \leq 1$. Then*

$$\frac{T_n^\circ}{n^\gamma} \rightarrow \mathcal{T} \quad \text{in probability, in the Gromov-Hausdorff sense,}$$

where \mathcal{T} is a γ -self-similar CRT with dislocation measure

$$\nu(ds) = \left(\lambda + (1 - 2\lambda) \sum_{i=1}^{\infty} s_i^2 \right) \text{PD}_{\alpha, \theta}^*(ds).$$

Regarding the alpha model, $\alpha \in (0, 1)$, $\theta = -2\alpha$, $\lambda = 1 - \alpha$, this confirms in part a conjecture formulated in [30]. Another interesting feature of the skewed Poisson-Dirichlet model relates to sampling consistency. Here we say that a family of unlabelled random trees $(T_n^\circ, n \geq 1)$ is *sampling consistent* if the tree T_n° with a uniformly chosen leaf removed is distributed as T_{n-1}° . For consistent trees with exchangeable labels such as the exchangeable Poisson-Dirichlet model this is trivially so, but also for the alpha-gamma model that includes non-exchangeable trees.

Proposition 34 *The skewed Poisson-Dirichlet model is sampling consistent only for parameters that reduce it to the exchangeable Poisson-Dirichlet model or to the alpha-gamma model.*

Proof. By Corollary 5, the skewed Poisson-Dirichlet model has dislocation measure

$$\kappa = \int_{S^1} (\lambda \kappa_{\mathbf{s}}(\cdot \cap \mathcal{P}^{\mathbf{0}[2]}) + (1 - \lambda) \kappa_{\mathbf{s}}(\cdot \cap \mathcal{P}^{\mathbf{1}[2]})) \text{PD}_{\alpha, \theta}^*(ds).$$

From this, we can calculate splitting rules. Specifically, we can calculate the distribution of the ranked sequence $S_n = (\#\Pi_{n,1}, \dots, \#\Pi_{n,K_n})^\downarrow$ of block sizes of $\Pi_n = (\Pi_{n,1}, \dots, \Pi_{n,K_n})$ by summing (4) over partitions of equal ranked sequence of block sizes and obtain

$$\begin{aligned} \mathbb{P}(S_2 = (1, 1)) &= 1, & \mathbb{P}(S_3 = (1, 1, 1)) &= \frac{\lambda(2\alpha + \theta)}{D_3}, & \mathbb{P}(S_3 = (2, 1)) &= \frac{(1 + \lambda)(1 - \alpha)}{D_3} \\ \mathbb{P}(S_4 = (1, 1, 1, 1)) &= \frac{\lambda(3\alpha + \theta)(2\alpha + \theta)}{D_4}, & \mathbb{P}(S_4 = (2, 1, 1)) &= \frac{(1 + 4\lambda)(2\alpha + \theta)(1 - \alpha)}{D_4} \\ \mathbb{P}(S_4 = (2, 2)) &= \frac{(1 + \lambda)(1 - \alpha)^2}{D_4} & \mathbb{P}(S_4 = (3, 1)) &= \frac{2(1 - \alpha)(2 - \alpha)}{D_4}, \end{aligned}$$

where D_3 and D_4 are normalisation constants of the form $a_3\lambda + b_3$ and $a_4\lambda + b_4$. Using the criterion of [20], sampling consistency requires, in particular, that

$$\mathbb{P}(S_3 = (1, 1, 1)) = \mathbb{P}(S_4 = (1, 1, 1, 1)) + \frac{1}{2}\mathbb{P}(S_4 = (2, 1, 1)) + \frac{1}{4}\mathbb{P}(S_4 = (3, 1))\mathbb{P}(S_3 = (1, 1, 1)),$$

which upon multiplication by D_3D_4 is a quadratic equation in λ . Common coefficients of all terms include $(1 - \alpha)$ and $(\theta + 2\alpha)$. For $\alpha < 1$ and $\theta > -2\alpha$, the quadratic equation has the two solutions $\lambda = 1/2$ and $\lambda = (1 - \alpha)/(1 - \theta - 2\alpha)$ corresponding, respectively, to the Poisson-Dirichlet and alpha-gamma models, so no other models can be sampling consistent.

The exchangeable Poisson-Dirichlet is trivially sampling consistent. The alpha-gamma model was shown in [11] to be sampling consistent. In the excluded case $\alpha = 1$ models for all θ collapse to the same deterministic model where all leaves are connected directly to a single branch point [26]. For the binary case $\theta = -2\alpha$, which we also had to exclude, we need to consider S_5 giving similar quadratic equations, but also lead to the required conclusion that only the alpha model $\lambda = 1 - \alpha$ and the beta-splitting model $\lambda = 1/2$ are sampling consistent. We leave the details to the reader. \square

A Proof of Lemma 27

The first part of Lemma 27 is a straightforward consequence of [16], see also [20, Lemma 8]. The second part generalises [20, Lemma 12]. For the convenience of the reader, we reproduce the proof here, rewritten for our higher generality.

Let $N_y(t_1, t_2)$ denote the number of jumps of ξ of size at least y in the time interval $[t_1, t_2]$, $\tilde{N}_y^{\varepsilon, \tau}(t_1, t_2)$ denote the number of jumps of $\exp(-\varepsilon)(1 - \exp(-\xi))$ of size at least y in the same time interval.

Step 1. Large deviations for $\tilde{N}_y^{\varepsilon, \tau}(0, \tau'')$.

Lemma 35 *For all $x > 0$ and $0 < y \leq 1$,*

$$\mathbb{P} \left(\tilde{N}_y^{\varepsilon, \tau}(0, \tau'') > (1+x)C_\Lambda \sum_{i=0}^{\lfloor \tau'' \rfloor} \exp(-\varrho(\varepsilon + \xi_i))\bar{\Lambda}(y) \right) \leq \exp(-a_x \bar{\Lambda}(y)),$$

where $a_x := (1+x)\ln(1+x) - x > 0$.

Proof. Let $\mathcal{F}_t^{\varepsilon, \tau}$ denote the σ -field generated by $(\varepsilon, \tau, \tau'' \wedge t)$ and ξ until time t , and $\mathcal{F}_\infty^{\varepsilon, \tau}$ the one generated by $(\varepsilon, \tau, \tau'')$ and ξ , and observe that

$$\tilde{N}_y^{\varepsilon, \tau}(0, \tau'') \leq \sum_{i=0}^{\lfloor \tau'' \rfloor} \tilde{N}_y^{\varepsilon, \tau}(i, i+1) \leq \sum_{i=0}^{\lfloor \tau'' \rfloor} N_{y \exp(\varepsilon + \xi_i)}(i, i+1).$$

Conditional on $\mathcal{F}_i^{\varepsilon, \tau}$, $N_{y \exp(\varepsilon + \xi_i)}(i, i+1)$ is a Poisson random variable with mean $\bar{\Lambda}(y \exp(\varepsilon + \xi_i))$. But for any Poisson random variables P with mean λ , one has

$$\mathbb{E}[\exp(\gamma P - (1+x)\gamma\lambda)] = \exp((\exp(t) - 1 - (1+x)\gamma)\lambda), \quad \forall \gamma \in \mathbb{R}.$$

In particular, when $\gamma = \ln(1+x)$, $\exp(\gamma) - 1 - (1+x)\gamma = -a_x < 0$ and the expectation is smaller than 1. Hence, for all $n \in \mathbb{N}$, using the tail bounds of Λ for the first inequality, we obtain, for all $y \leq 1$,

$$\begin{aligned} & \mathbb{P} \left(\sum_{i=0}^{\lfloor \tau'' \rfloor \wedge n} N_{y \exp(\varepsilon + \xi_i)}(i, i+1) \geq (1+x)C_\Lambda \sum_{i=0}^{\lfloor \tau'' \rfloor \wedge n} \exp(-\varrho(\varepsilon + \xi_i))\bar{\Lambda}(y) \right) \\ & \leq \mathbb{P} \left(\sum_{i=0}^{\lfloor \tau'' \rfloor \wedge n} N_{y \exp(\varepsilon + \xi_i)}(i, i+1) \geq (1+x) \sum_{i=0}^{\lfloor \tau'' \rfloor \wedge n} \bar{\Lambda}(y \exp(\varepsilon + \xi_i)) \right) \\ & \leq \mathbb{E} \left[\exp \left(\gamma \left(\sum_{i=0}^{\lfloor \tau'' \rfloor \wedge n} (N_{y \exp(\varepsilon + \xi_i)}(i, i+1) - (1+x)\bar{\Lambda}(y \exp(\varepsilon + \xi_i))) \right) \right) \right] \\ & \leq \mathbb{E} \left[\exp \left(\gamma \left(\sum_{i=0}^{\lfloor \tau'' \rfloor \wedge (n-1)} \dots \right) \right) \mathbb{E}[\exp(\gamma 1_{\{\tau'' \geq n\}}(N_{y \exp(\varepsilon + \xi_n)}(n, n+1) \right. \\ & \qquad \qquad \qquad \left. - (1+x)\bar{\Lambda}(y \exp(\varepsilon + \xi_n)))) | \mathcal{F}_n^{\varepsilon, \tau}] \right] \\ & \leq \dots \leq \exp(-a_x \bar{\Lambda}(y)), \end{aligned}$$

the last line being obtained by induction: at each step but the last we use the upper bound 1 for the conditional expectation and for the last step, we use the upper bound $\exp(-a_x \bar{\Lambda}(y))$ for the expectation $\mathbb{E}[\exp(\gamma(N_y(0, 1) - (1+x)\bar{\Lambda}(y)))]$. It remains to let $n \rightarrow \infty$ in the first probability involved in the above sequence of inequalities and to use Fatou's lemma. \square

Step 2. Large deviations for $\mathbb{E}[K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}]$.

Lemma 36 *Let $B_\alpha := \sum_{k=1}^{\infty} \exp(-4^{-1}a_1 k^{\alpha/2})$ with $a_1 = 2 \ln 2 - 1$. Then for all $x \geq 1$ and all integers n large enough,*

$$\mathbb{P} \left(\mathbb{E} [K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] > (1+x)(Y(\varepsilon, \tau, \tau') - 1)\bar{\Lambda}(n^{-1}) \right) \leq (1+B_\alpha) \exp(-4^{-1}a_1 x \bar{\Lambda}(n^{-1})).$$

Proof. According to formula (4) of [16],

$$\mathbb{E} [K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] = n \int_0^1 (1-y)^{n-1} \tilde{N}_y^{\varepsilon, \tau}(0, \tau'') dy \leq \tilde{N}_{1/n}^{\varepsilon, \tau}(0, \tau'') + n \int_0^{1/n} \tilde{N}_y^{\varepsilon, \tau}(0, \tau'') dy.$$

Hence, setting $S := C_\Lambda \sum_{i=0}^{\lceil \tau'' \rceil} \exp(-\varrho(\varepsilon + \xi_i))$,

$$\begin{aligned} & \mathbb{P} \left(\mathbb{E} [K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] > (1+x)(1+A_\alpha)S\bar{\Lambda}(n^{-1}) \right) \\ & \leq \mathbb{P} \left(\tilde{N}_{1/n}^{\varepsilon, \tau}(0, \tau'') > (1+x)S\bar{\Lambda}(n^{-1}) \right) + \mathbb{P} \left(n \int_0^{1/n} \tilde{N}_y^{\varepsilon, \tau}(0, \tau'') dy > (1+x)A_\alpha S\bar{\Lambda}(n^{-1}) \right). \end{aligned}$$

The first probability in the right-hand side is smaller than $\exp(-a_x \bar{\Lambda}(n^{-1}))$ by Lemma 35. To bound the second probability, we use $n \int_{1/(k+1)n}^{1/kn} \tilde{N}_y^{\varepsilon, \tau}(0, \tau'') dy \leq \tilde{N}_{1/(n(k+1))}^{\varepsilon, \tau}(0, \tau'') \frac{1}{k(k+1)}$, which gives

$$\begin{aligned} & \mathbb{P} \left(n \int_0^{1/n} \tilde{N}_y^{\varepsilon, \tau}(0, \tau'') dy > A_\alpha(1+x)S\bar{\Lambda}(n^{-1}) \right) \\ & \leq \sum_{k=1}^{\infty} \mathbb{P} \left(\tilde{N}_{1/(n(k+1))}^{\varepsilon, \tau}(0, \tau'') > 2(k+1)^{\sqrt{\alpha}}(1+x)S\bar{\Lambda}(n^{-1}) \right). \end{aligned}$$

Since $\bar{\Lambda}$ is regularly varying at 0 with index $-\alpha$, we have, provided n is large enough, that $\bar{\Lambda}(n^{-1})(k+1)^{\alpha/2} \leq 2\bar{\Lambda}((n(k+1))^{-1})$ and $\bar{\Lambda}(((k+1)n)^{-1}) \leq 2\bar{\Lambda}(n^{-1})(k+1)^{\sqrt{\alpha}}$ for all $k \geq 1$ (to see this, use, e.g. Potter's theorem, Theorem 1.5.6, [10]). Combined with Lemma 35, this implies that the above sum of probabilities is smaller than

$$\sum_{k=1}^{\infty} \exp(-a_x \bar{\Lambda}((n(k+1))^{-1})) \leq \sum_{k=1}^{\infty} \exp(-2^{-1}a_x \bar{\Lambda}(n^{-1})(k+1)^{\alpha/2}).$$

Last, the exponential in the latter sum can be split in two, using $(k+1)^{\alpha/2} \geq 2^{-1}(k^{\alpha/2} + 1)$, to get the upper bound

$$\exp(-4^{-1}a_x \bar{\Lambda}(n^{-1})) \sum_{k=1}^{\infty} \exp(-a_x 4^{-1} \bar{\Lambda}(n^{-1}) k^{\alpha/2}),$$

which is smaller than $\exp(-a_1 x \bar{\Lambda}(n^{-1}))B_\alpha$ for all $x \geq 1$ ($a_x \geq a_1 x$ for $x \geq 1$) and n large enough. \square

Step 3. Proof of inequality (19). To start with, fix $x \geq 1$, $n \in \mathbb{N}$, and note that

$$\begin{aligned} & \mathbb{P} (K_n(\varepsilon, \tau, \tau') > (1+x)Y(\varepsilon, \tau, \tau')\bar{\Lambda}(n^{-1})) \\ & \leq \mathbb{P} \left(\mathbb{E} [K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] > (1+x)(Y(\varepsilon, \tau, \tau') - 1)\bar{\Lambda}(n^{-1}) \right) \\ & \quad + \mathbb{P} (K_n(\varepsilon, \tau, \tau') - \mathbb{E} [K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] > (1+x)\bar{\Lambda}(n^{-1})). \end{aligned} \tag{24}$$

Lemma 36 gives an upper bound for the first probability provided n is large enough. To get an upper bound for the second probability, we use a result on urn models (Devroye [12], Section 6) which ensures that

$$\mathbb{P}(K_n(\varepsilon, \tau, \tau') - \mathbb{E}[K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] > y | \mathcal{F}_{\tau''}^{\varepsilon, \tau}) \leq \exp\left(-\frac{y^2}{2\mathbb{E}[K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] + 2y/3}\right)$$

for all $y \geq 0$, $n \in \mathbb{N}$. This implies that for all $m \geq 1$, there exists some deterministic constant B_m depending only on m such that

$$\begin{aligned} & \mathbb{P}(K_n(\varepsilon, \tau, \tau') - \mathbb{E}[K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] > (1+x)\bar{\Lambda}(n^{-1}) | \mathcal{F}_{\tau''}^{\varepsilon, \tau}) \\ & \leq B_m \left(\frac{\mathbb{E}[K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] + (1+x)\bar{\Lambda}(n^{-1})}{((1+x)\bar{\Lambda}(n^{-1}))^2} \right)^m \\ & \leq 2^{m-1} B_m \frac{(\mathbb{E}[K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}])^m + ((1+x)\bar{\Lambda}(n^{-1}))^m}{((1+x)\bar{\Lambda}(n^{-1}))^{2m}} \\ & \leq 2^{m-1} B_m \frac{\mathbb{E}[(K_n(\varepsilon, \tau, \tau'))^m | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] + ((1+x)\bar{\Lambda}(n^{-1}))^m}{((1+x)\bar{\Lambda}(n^{-1}))^{2m}}, \end{aligned}$$

the last line being obtained by Jensen's inequality. We then take expectations on both sides of the resulting inequality. Theorem 6.3 of [16] ensures that $\mathbb{E}[(K_n(\varepsilon, \tau, \tau'))^m | \varepsilon, \tau] \leq \mathbb{E}[(K_n(0, 0, \infty))^m] \sim (\bar{\Lambda}(n^{-1}))^m$ (up to a constant). Therefore, we have

$$\mathbb{P}(K_n(\varepsilon, \tau, \tau') - \mathbb{E}[K_n(\varepsilon, \tau, \tau') | \mathcal{F}_{\tau''}^{\varepsilon, \tau}] > (1+x)\bar{\Lambda}(n^{-1})) \leq B_{m,\Lambda} ((1+x)\bar{\Lambda}(n^{-1}))^{-m}, \quad (25)$$

where $B_{m,\Lambda}$ depends only on m and Λ .

Next, recall the upper bound given by Lemma 36 for the first probability involved in the right-hand side of (24). Together with the upper bound (25), it leads to the existence of $B'_{m,\Lambda}$ such that

$$\mathbb{P}(K_n(\varepsilon, \tau, \tau') > (1+x)Y(\varepsilon, \tau, \tau')\bar{\Lambda}(n^{-1})) \leq B'_{m,\Lambda} x^{-p} (\bar{\Lambda}(n^{-1}))^{-m}$$

for all $x \geq 1$ and n large enough, say $n \geq n_0$. Since $\bar{\Lambda}(n^{-1}) \sim n^\alpha \ell(n)$ when $n \rightarrow \infty$, this upper bound is in turn bounded from above by $x^{-m} n^{1-\alpha m}$ up to some constant, which is the required result (19).

Finally, inequality (19) is also true when $n \leq n_0$ (for all $x \geq 1$), since $K_n(\varepsilon, \tau, \tau') \leq n \leq n_0$ and $Y(\varepsilon, \tau, \tau') \geq 1$, and therefore the probability $\mathbb{P}(K_n(\varepsilon, \tau, \tau') > (1+x)Y(\varepsilon, \tau, \tau')\bar{\Lambda}(n^{-1}))$ is null whenever $1+x \geq n_0 (\bar{\Lambda}(n^{-1}))^{-1}$.

This completes the proof of Lemma 27. \square

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