4 Life insurance

Please hand in your answers to questions 1, 2(ii), 5, 7, 9, 10, 13, 16 and 19 for marking. The rest are for further practice.

1. (i) Show algebraically that $A_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$. Also demonstrate the result verbally.

   (ii) Show algebraically that $(IA)_x = \ddot{a}_x - d(I\ddot{a})_x$. Demonstrate the result verbally. [*Here $(IA)_x$ is the expected time-0 value (assuming an age of x at time 0) of a single payment of size $K+1$ made at time $K+1$, where death occurs in the year $(K, K+1)$. $(I\ddot{a})_x$ is the expected value of the stream of payments of 1 at time 0, 2 at time 1, ..., $K+1$ at time K, with the last payment made at the beginning of the year of death.*]

2. Suppose that the future lifetime of a life aged x ($x > 0$) is represented by a random variable T_x distributed on the interval $(0, \omega - x)$, where ω is some maximum age.

   (i) For $0 \leq x < y < \omega$, state a consistency condition between the distributions of T_x and of T_y.

   (ii) Suppose that $0 \leq x < y < \omega$, and let $t = y - x$. Define the force of mortality at age y: (a) in terms of T_0; and (b) in terms of T_x, and show that the two definitions are equivalent.

   (iii) Prove that ${}_tp_x = \exp\left\{ -\int_0^t \mu_{x+s} ds \right\}$.

3. Given a curtate lifetime $K = [T]$ and a constant-δ interest model, consider the following insurance products:

   (i) pure endowment with term n;

   (ii) whole life assurance;

   (iii) term assurance with term n;

   (iv) endowment assurance with term n (under this product, there is a payment of one unit at the end of the year of death or at end of term whichever is earlier).

   (a) Write each one as a random cash-flow of the form $C = \big((t_k, c_k B_k)\big)_{k=1,2,\ldots}$, where the B_k are Bernoulli random variables defined in terms of K.

   (b) Find expressions for the net premiums and variances of these products.

   (c) Relate the products, premiums and variances of (iv) and (i) and (iii).

   (d) Comment on (a) and (b) for the corresponding products where payment is made at death rather than at the end of the year of death (and n is not necessarily an integer).

4. A cash-flow is payable continuously at a rate of $\rho(t)$ per annum at time t provided a life who is aged x at time 0 is still alive. T_x is a random variable which models the residual lifetime in years of a life aged x.

   (a) Write down an expression, in terms of T_x, for the (random) present value at time 0 of this cash-flow, at a constant force of interest δ p.a., and show that the expected present value at time 0 of the cash-flow is equal to

   $$\int_0^\infty e^{-\delta s} \rho(s) \mathbb{P}(T_x > s) ds.$$

(b) An annuity is payable continuously during the lifetime of a life now aged 30, but for at most 10 years. The rate of payment at all times t during the first 5 years is £5,000 p.a., and thereafter £10,000 p.a. The force of mortality to which this life is subject is assumed to be 0.01 p.a. at all ages between 30 and 35, and 0.02 p.a. between 35 and 40. Find the expected present value of this annuity at a force of interest of 0.05 p.a.

(c) If the mortality and interest assumptions are as in (b), find the expected present value of the benefits of a term assurance, issued to the life in (ii), which pays £40,000 immediately on death within 10 years.

For the questions below, use the A 1967-70 Mortality table whenever table data is needed (all three columns, assuming medical checks were successful).

5. Find the present value of £5000 due in 5 years' time at $i = 4\%$ if

   a) the payment is certain to be made;

   b) the payment is contingent upon a life aged 35 now surviving to age 40.

6. How large a pure endowment, payable at age 65 can a life aged 60 buy with £1000 cash if $i = 7\%$?

7. a) Show that $\ddot{a}_x = 1 + (1+i)^{-1} p_x \ddot{a}_{x+1}$

   b) Show that $\ddot{a}_{x:\overline{n}|} = a_{x:\overline{n}|} + 1 - A_{\substack{1 \\ x:\overline{n}|}}$

   c) Show that $A_x = (1+i)^{-1}\ddot{a}_x - a_x$

8. Find the net single premium for a 5-year temporary life annuity issued to a life aged 65 if $i = 8\%$ for the first 3 years and $i = 6\%$ for the next 2 years.

9. Find the net annual premium for a £40000, 4-year term assurance policy issued to a life aged 26 if $i = 10\%$.

10. A deferred (temporary) life annuity is a deferred perpetuity (annuity-certain) restricted to a lifetime. For a life aged x, deferred period of m years (and a term of n years, from $m+1$ to $m+n$), the notation for the single premium is ${}_{m|}a_x$ (respectively ${}_{m|}a_{x:\overline{n}|}$).

    a) Give expressions for ${}_{m|}a_x$ and ${}_{m|}a_{x:\overline{n}|}$ in terms of the life table probabilities q_k.

    b) Show that ${}_{m|}a_x = A_{\substack{1 \\ x:\overline{m}|}} a_{x+m}$.

11. Describe the benefit which has the present value random variable function given by Z below; T denotes the future lifetime of a life aged x.

$$Z = \begin{cases} \bar{a}_{\overline{n}|} & T \leq n \\ \bar{a}_{\overline{T}|} & T > n \end{cases}$$

12. A special deferred annuity provides as benefits for a life aged 60:

    - on survival to age 65 an annuity of £2,000 p.a. payable in advance for two years certain and for life thereafter

    - on death between ages 63 and 65, £5,000 payable at the end of the year of death

    - on death between ages 60 and 63, £10,000 payable at the end of the year of death.

Annual premiums are payable in advance until age 65 or earlier death. Determine the level annual premium based on an effective interest rate of 4% p.a. For the life annuity part, you may approximate all one-year death probabilities for age greater than 65 by the constant 0.05 (which leads close to the correct numerical answer).

13. Give an expression (in terms of standard actuarial functions) for the annual premium for a 25 year endowment assurance on a life aged 40. The initial expenses are £2 per £100 sum insured, renewal expenses are 5% of each premium and 30p each year per £100 sum insured, and there is an initial commission of 50% of the first year's gross premium.

14. Some time ago, a life office issued an assurance policy to a life now aged exactly 55. Premiums are payable annually in advance, and death benefits are paid at the end of the year of death. The office calculates reserves using gross premium policy values. The following information gives the reserve assumptions for the policy year just completed. Expenses are assumed to be incurred at the start of the policy year.

    Reserve brought forward at the start of the policy year: £12,500

    Annual premium: £1,150

    Annual expenses: £75

    Death benefit: £50,000

    Mortality: A1967/70

    Interest 5.5% per annum

    Calculate the reserve at the end of the policy year.

15. A deferred annuity is purchased by 20 annual payments payable by a life aged 40 for a year annuity in advance of £2,500 a year, commencing in 20 years, for life. Find an expression for the premium on the basis of 4% pa interest with expenses of 5% of each premium and £5 at each annuity payment.

16. Suppose that $l_x = 100,000(100 - x)$, where $0 \leq x \leq 100$, and the interest rate is 5%.

    (a) Calculate $A_{50:\overline{10}|}$ and $\ddot{a}_{50:\overline{10}|}$.

    (b) Calculate the net annual premium for a 10 year endowment assurance for £10,000 to someone aged 50 and the policy values of years 3 and 4 using the values above.

    (c) Suppose that expenses are as follows

Commission:	50% of First Premium
	2% of Subsequent Premiums
General Expenses:	£150 Initially
	£10 in each subsequent year.

    Calculate the office premium for the policy in (b).

17. Show algebraically that the product of

    - the reserve after t years for an annual premium n-year pure endowment issued to a life aged x

    and

- the annual premium for pure endowment of like amount issued at the same age but maturing in t years

is constant for all values of t. (Ignore expenses.)

Try to find an argument for this by general reasoning also.

18. (i) Show that $\overline{A}_x = 1 - \delta\overline{a}_x$.

    (ii) Consider a whole life assurance with sum assured 1 payable at the point of death, with a constant premium paid continuously. Show that the reserve at time t satisfies

$$_t\overline{V}_x = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x}$$

19. Thiele's equation for the policy value at duration t, $_t\overline{V}_x$, of a life annuity payable continuously at rate 1 from age x is:

$$\frac{\partial}{\partial t}{}_t\overline{V}_x = \mu_{x+t}\ {}_t\overline{V}_x - 1 + \delta\ {}_t\overline{V}_x.$$

Derive this result algebraically, and also explain it by general reasoning.

20. Let $_tV_x$ be the reserve at time t on an insurance policy issued to a life aged x that pays a benefit of $S_x(t)$ on death at time t, under which premiums are payable of $P(t)$ at time t. Show by general reasoning that

$$\frac{\partial}{\partial t}{}_tV_x = P(t) + \delta\ {}_tV_x - \mu_{x+t}(S_x(t) - {}_tV_x)$$

where δ is the force of interest.

21. An insurer issues n identical policies. Let Y_j be the claim amount from the jth policy, and suppose that the random variables Y_j, $j = 1, \ldots, n$ are i.i.d. with mean $\mu > 0$ and variance σ^2. The insurer charges a premium of A for each policy.

    (a) Show that if $A = \mu + 10\sigma n^{-1/2}$, then the probability that total claims exceed total premiums is no more than 1%, for any value of n.

    (b) Use the Central Limit Theorem to show that if instead $A = \mu + 3\sigma n^{-1/2}$, then this probability is still less than 1%, provided n is large enough.