1 Cash-flows, discounting, interest rates and yields

Please hand in your answers to questions 3, 4, 5, 8, 11 and 12 for marking. The rest are for further practice.

1. Discuss the cash-flows of a typical final-year undergraduate over 5 years. Aim for a concise presentation.

2. If the effective annual rate of interest is 9%, calculate the total accumulated value at 1 January 2010 of payments of £100 made on 1 January 2009, 1 April 2009, 1 July 2009 and 1 October 2009.

3. A man stipulates in his will that £50,000 from his estate is to be placed in a fund from which his three children are each to receive an equal amount when they reach age 21. When the man dies, the children are ages 19, 15 and 13. If this fund earns 6% interest per half-year (i.e. nominal 12% p.a. compounded each 6 months), what is the amount that they each receive? Is the distribution fair?

4. (i) Establish a table of relationships between (constant) δ, i, v and d.
   (ii) Show that $d = vi$ and interpret this.
   (iii) Show that $\delta \approx i - i^2/2$ and $d \approx i - i^2$ for small i, and that $d \approx \delta - \delta^2/2$ for small δ.

5. Calculate the equivalent effective annual rate of interest of
   (a) a force of interest of 7.5% p.a.
   (b) a discount rate of 9% p.a.
   (c) a nominal rate of interest of 8% p.a. convertible half-yearly
   (d) a nominal rate of interest of 9% p.a. convertible monthly.

6. A borrower is under an obligation to repay a bank £6,280 in four years' time, £8,460 in seven years' time and £7,350 in thirteen years' time.

   As part of a review of his future commitments the borrower now offers either

   (a) to discharge his liability for these three debts by making an appropriate single payment five years from now; or
   (b) to repay the total amount owed (i.e. £22,090) in a single payment at an appropriate future time.

   On the basis of a constant interest rate of 8% per annum, find the appropriate single payment if offer (a) is accepted by the bank, and the appropriate time to repay the entire indebtedness if offer (b) is accepted. (You should work on the basis that the present value of the single payment under the revised arrangement should equal the present value of the three payments due under the current obligation).

7. Calculate the accumulated value of £1,000 after 2 years if $\delta(t) = 0.06(t+1)$ for $0 \le t \le 2$.

8. *Stoodley's formula.* Suppose the force of interest is given by

$$\delta(t) = p + \frac{s}{1 + re^{st}}, \qquad t \in \mathbb{R}_+$$

where $p, r \in \mathbb{R}_+$ and $s \geq -p$.

Calculate the discount factor $v(t)$, $t \in \mathbb{R}_+$, and show that the model can be reparametrized such that $v(t) = \lambda v_1^t + (1-\lambda)v_2^t$. Interpret this!

9. In valuing future payments an investor uses the formula

$$v(t) = \frac{\alpha(\alpha+1)}{(\alpha+t)(\alpha+t+1)}, \qquad t \in \mathbb{R}_+$$

where α is a given positive constant, for the value at time 0 of 1 due at time t (measured in years).

Show that the above formula implies that

(i) the force of interest per annum at time t will be

$$\delta(t) = \frac{2t+2\alpha+1}{(\alpha+t)(\alpha+t+1)}$$

(ii) the effective rate of interest for the period r to $r+1$ will be

$$i(r) = \frac{2}{r+\alpha}$$

(iii) the present value of a series of n payments, each of amount 1 (the rth payment being due at time r) is

$$a(n) = \frac{n\alpha}{n+\alpha+1}.$$

(iv) Suppose now that $\alpha = 15$. Find the level annual premium, payable in advance for twelve years, which will provide an annuity of £1,800 per annum, payable annually for ten years, the first annuity payment being made one year after payment of the final premium. What is the value at time 12 of the series of annuity payments, what at time 0?

10. Under the terms of a savings scheme an investor who makes an initial investment of £4,000 may receive either

- £2,000 after 2 years and a further £3,000 after 7 years; or
- £4,400 at the end of 4 years.

Which of these options corresponds to a higher rate of interest on the investor's money?

11. Find the yield of an investment of X at time s, followed by a single payment of Y received at time $t > s$. How does the yield change as X, Y, s, t vary?

12. Find the yield of a bond costing P at time 0, which generates coupon payments of X at times $1, 2, \ldots, n$ and a redemption payment of P at time n.

Course webpage, where all other assignments and solutions will be posted:
`http://www.stats.ox.ac.uk/~winkel/sb4a.html`

Class allocations and hand-in times:
`https://minerva.stats.ox.ac.uk/perl/classlists.pl`

2 $p$thly payments, annuities, loans and funds

Please hand in your answers to questions 1, 4, 5, 6, 9 and 12 for marking. The rest are for further practice.

1. For $m \in \mathbb{N}$, the prefix $_{m|}$ before an annuity symbol indicates that the sequence of payments concerned is *deferred* by an amount of time m. For example, the discounted present value (in the constant interest-rate model) of a *deferred annuity*, with unit payments per unit time payable from $m+1$ to $m+n$, is denoted by $_{m|}a_{\overline{n}|}$.

   (a) Express $_{m|}a_{\overline{n}|}$ in terms of the ordinary annuity symbols introduced in the lectures.

   (b) The case $m = -1$ corresponds to an annuity-due and is denoted by $\ddot{a}_{\overline{n}|}$. Express $\ddot{a}_{\overline{n}|}$ as simply as you can (i) in terms of $a_{\overline{n}|}$ and (ii) in terms of $a_{\overline{n-1}|}$

   (c) The discounted present value of an *increasing annuity* with payments j at time $j = 1, \ldots, n$ is denoted by $(Ia)_{\overline{n}|}$. Express $(Ia)_{\overline{n}|}$ in terms of ordinary annuity symbols.

   (d) Consider a security redeemable at par, with term n and $p$thly coupon payments at nominal rate j. Express the accumulated (time n) and discounted (time 0) values in the constant i model in terms of annuity symbols.

2. Find, on the basis of an effective interest rate of 4% per unit time, the values of

$$a^{(4)}_{\overline{67}|}, \quad s^{(12)}_{\overline{18}|}, \quad \ddot{a}^{(4)}_{\overline{16.5}|}, \quad \ddot{s}^{(12)}_{\overline{15.25}|}, \quad {}_{4.25|}a^{(4)}_{\overline{3.75}|}, \quad (Is)_{\overline{4}|}.$$

   Describe the meaning of each of the symbols.

3. Buy now, pay later!! A luxury sofa is sold with the following payment options: either pay immediately and receive a 5% discount, or pay by monthly instalments (in arrears) for 15 months. To calculate the monthly payments, add a 5% finance charge to the purchase price, then divide the total amount into 15 equal payments. What annual effective rate of interest is being charged for paying by instalments?

4. Show that in any constant i interest rate model, the cash-flows $c_\infty(s) = \delta$, $0 \leq s \leq 1$, and $c_p = (k/p, i^{(p)}/p)_{k=1,\ldots,p}$ are equivalent (have the same value) for all $p \in \mathbb{N}$.

5. A loan of £30,000 is to be repaid by a level annuity payable monthly in arrears for 25 years, and calculated on the basis of an (effective) interest rate of 12% pa. Calculate the intitial monthly repayments.

   After ten years of repayments the borrower asks to:

   (a) pay off the loan which is outstanding. Calculate the lump sum which would be required to pay off the outstanding loan.

   (b) extend the loan by a further five years (i.e. to 30 years in total), and with repayments changed from monthly to quarterly in arrears. Calculate the revised level of quarterly repayments.

   (c) reduce the loan period by five years (i.e. to 20 years in total) and for repayments to be biannually (i.e. once every two years) in arrears. Calculate the revised level biannual repayments.

   [*Hint: calculate annual repayment levels first, then combine each two consecutive payments into an equivalent single payment.*]

6. An insurance company issues an annuity of £10,000 p.a. payable monthly in arrears for 25 years. The cost of the annuity is calculated using an effective rate of 10% p.a.
   (a) Calculate the interest component of the first instalment of the sixth year.
   (b) Calculate the total interest paid in the first 5 years.
7. A 10-year loan for £10,000 has a fixed rate of 5% for the first 3 years. Thereafter the rate is 8% for the rest of the term. Repayments are annual. Find the amounts (i) of the first 3 payments and (ii) of the remaining 7 payments. Show that the APR of the loan is 6.4%.
8. (a) An annuity-certain is payable annually in advance for n years. The first payment of the annuity is 1. Thereafter the amount of each payment is $(1+r)$ times that of the preceding payment. Show that, on the basis of an interest rate of i per annum, the present value of the annuity is $\ddot{a}_{\overline{n}|j}$ where $j = (i-r)/(1+r)$.
   (b) Suppose instead that the annuity is payable annually in arrear. Is its present value (at rate i) now equal to $a_{\overline{n}|j}$?
   (c) In return for a single premium of £10,000 an investor will receive an annuity payable annually in arrear for 20 years. The annuity payments increase from year to year at the (compound) rate of 5% per annum. Given that the initial amount of the annuity is determined on the basis of an interest rate of 9% per annum, find the amount of the first payment.
9. A couple buying a house require a £150,000 mortgage. They know that they will have to sell the house (and redeem the mortgage) in three years' time. They have the choice of two mortgages with monthly payments and 25-year term.

   Mortgage A is at effective rate $i_A = 10.25\%$ p.a. This mortgage stipulates, however, that if you pay off the mortgage any time before the fifth anniversary, you will have to pay a penalty equal to 2.4% of the outstanding debt at the time of repayment.

   Mortgage B is at $i_B = 10.7\%$ but can be paid off at any time without penalty.

   Given that the buyers will have to repay the mortgage after three years and that they can save money at $j = 6\%$, which mortgage should they choose?

   Calculate the yields (assuming redemption after three years).
10. *Hardy's Formula.* Consider a fund F, and incomplete information as follows: the amounts of the fund $F_0 \in \mathbb{R}_+$ just before time 0 and $F_1 \in \mathbb{R}_+$ just after time 1 and the (net) total new money $N \in \mathbb{R}$ invested during $[0, 1]$. Denote $I = F_1 - F_0 - N$, the total interest and capital gain earned by the fund during $[0, 1]$.

    (a) Assume that N is received in two equal instalments at the beginning and end of the year. Show that the money-weighted rate of return is given by the formula

    $$i = \frac{2I}{F_1 + F_0 - I}.$$

    What is the time-weighted rate of return?
    (b) Assume that N is received in one instalment at $t = 1/2$. Find an expression for the MWRR. What about the TWRR?
    (c) Assume that N is received continuously at constant rate in $[0, 1]$. Give the yield equation for the money-weighted rate of return i. Show that for small i, the formula in part (a) is still a good approximation for the MWRR.

11. (i) A fund had the following revenue account for 2004:

		£million
Value of fund at 1 January 2004		30
Add: new investments received during the year	18	
Deduct: withdrawals and other payments	30	
Value of fund at 31 December 2004		21

Assuming the new investments received and the withdrawals and other payments are spread evenly over the year, use Hardy's Formula from the previous question to caluclate an approximate effective annual rate of interest for 2004.

(ii) A second set of accounts was constructed to show the dates at which the new investments were received, the dates at which the withdrawals and other payments actually occurred, and the value of the fund at various times during the year:

		£million
Value of fund at 1 January 2004		30
Value of fund at 31 March		36
Value of fund at 15 May		35
New investments at 16 May	18	
Value of fund at 30 June		51
Value of fund at 30 September		45
Withdrawals on 1 October	30	
Value of fund at 31 December 2004		21

(a) Show that the annual yield (i.e. the annual money weighted rate of return) on the fund for the year ending 31 December 2004 is approximately 8.87%.

(b) Calculate the annual TWRR on the fund for the year ending 31 Dec 2004.

(iii) Explain the differences between your answers for (i), (ii)(a) and (ii)(b).

12. In a particular accumulation fund income is retained and used to increase the value of the fund unit. The 'middle price' of the unit on 1 April in each of the years 1999 to 2005 is given in the following table:

Year 1 April	1999	2000	2001	2002	2003	2004	2005
Middle price of unit in £	1.86	2.11	2.55	2.49	2.88	3.18	3.52

(a) On the basis of the above prices and ignoring taxation and expenses:

(i) Find the time-weighted rate of return over the period 1 April 1999 to 1 April 2005,

(ii) Show that the yield obtained by an investor who purchased 200 units on 1 April in each year from 1999 to 2004 inclusive, and who sold his holding on 1 April 2005, is approximately 10.60%.

(iii) Show that the yield obtained by a person who invested £500 in the fund on 1 April each year from 1999 to 2004 inclusive, and who sold back his holding to the fund managers on 1 April 2005, is approximately 10.67% (You should assume that investors may purchase fractional parts of units.)

(b) Suppose that, in order to allow for expenses, the fund's managers sell units 2% above the published middle price and buy back units 2% below the middle price. On this basis find revised answers to (ii) and (iii) of (a).

3 Inflation, taxation, project appraisal, lifetime distributions

Please hand in your answers to questions 1, 2, 5, 6, 7, 8 and 10 for marking. The rest are for further practice.

1. A loan of £25,000 was issued and was repaid at par after three years. Interest was paid on the loan at the rate of 8% per annum, payable annually in arrears. The value of the Retail Price Index at various times was as follows

At issue	One year later	Two years later	Three years later
205.0	215.6	223.5	231.5

Calculate the real rate of return of the loan.

2. Calculate the time-dependent force of inflation between two successive index values if intermediate values are computed by (a) linear (b) exponential interpolation.

3. On 15 May 1997 the government of a country issued an index-linked bond of term 15 years. Coupons are payable half-yearly in arrears, and the annual nominal coupon rate is $D = 4\%$.

   Interest and capital repayments are indexed by reference to the value of a retail price index with a time lag of 8 months. The retail price index value in September 1996 was $Q(-8/12) = 200$ and in March 1997 was $Q(-2/12) = 206$.

   The issue price of the bond was such that, if the retail price index Q were to increase continuously at a rate of $e_1 = 7\%$ p.a. from March 1997, a tax exempt purchaser of the bond at the issue date would obtain a real yield of $y = 3\%$ p.a. convertible half-yearly.

   (i) Determine $Q(t)$ for all $t \geq 0$ and hence the coupon and redemption payments of the bond per 100 nominal, assuming inflation at constant rate e_1.

   (ii) Derive the formula for the price of the bond at issue to a tax-exempt investor and show that the issue price of the bond is £111.53 per 100 nominal.

   (iii) An investor purchases a bond at the price calculated in (ii) and holds it to redemption. The actual rate of increase of the retail price index is $e_2 = 5\%$ p.a. from March 1997. A new tax is introduced such that the investor pays tax at 40% on any real capital gain, where the real capital gain is the difference between the redemption money and the purchase price revalued according to the retail price index to the redemption date. Tax is only due if the real capital gain is positive. Calculate the real annual yield convertible half-yearly actually obtained by the investor.

4. A zero-coupon bond was purchased m years ago by investor A who is liable to capital gains tax at rate t. At the time of purchase the outstanding term of the bond was n years ($n > m$). The price paid by A will provide him with a net effective annual yield of $i > 0$ if he holds the bond until it is redeemed.

   Investor A now wishes to sell the bond. He will be liable to capital gains tax on the excess of his selling price over his purchase price.

   (a) Derive an expression in terms of t, n, and i for the purchase price (per unit redemption money) paid by A.

(b) Derive also an expression in terms of t, n, m, and i for the price (per unit redemption money) at which A should now sell the bond in order to obtain a net annual yield of i on the completed transaction.

(c) Assume that in fact the bond is sold by A to a second investor, who is also liable to capital gains tax at rate t, at a price which will provide the *new* purchaser with a net annual yield of i, if he holds the bond until it is redeemed.

Derive an equation from which can be found the value of j, the net annual yield obtained by A on the completed transaction.

Find the value of j when $n = 10$, $m = 5$, $t = 0.4$, and $i = 0.1$.

5. A bond pays coupons twice yearly in arrears at nominal annual rate $j^{(2)} = 5\%$, and will be redeemed at par after 5 years.

   An investor will be liable to capital gains tax at 40% on the difference between redemption price and purchase price, adjusted for inflation over the 5-year period.

   Under the assumption of a constant inflation rate of 2% p.a., find the purchase price which provides the investor with a yield (after capital gains tax) of (i) 6% (ii) 8%. What is the corresponding *real* yield in each case?

6. An investor purchases a bond 3 months after issue. The bond will be redeemed at par ten years after issue and pays coupons of 6% p.a. annually in arrears. The investor pays tax of 25% on both income and capital gains (no relief for indexation).

   (a) Calculate the purchase price of the bond per £100 nominal to provide the investor with a rate of return of 8% per annum effective.

   (b) The real rate of return expected by the investor from the bond is 3% per annum effective. Calculate the annual rate of inflation expected by the investor.

7. Two business projects, each of which takes two years to complete, produce the following cash-flows:

Project A:	Project B
• initial income of £2000;	• initial income of £360;
• after one year, expenditure of £3900;	• after one year, expenditure of £4000;
• after two years, income of £2000;	• after two years, income of £4000.

   An investor considering the two projects has no spare cash, but can borrow or invest money at rate $i > 0$ for any desired term. For what range of i is each of the projects profitable? What can you say about the yields of the two projects?

   If both projects are profitable and the investor must choose between them, which is the more profitable (for various possible values of i)?

8. An investor has decided to purchase a leasehold property for £80,000, with a further payment of £5000 for repairs in one year's time. The income associated with letting the property will be £10,000 per annum, payable continuously for 20 years commencing in two years' time.

   (a) i. Given that the venture will be financed by bank loans on the basis of an effective annual interest rate of 7% and that the loans may be repaid continuously, find the discounted payback period for the project.

   ii. Given, further, that after the loans have been repaid the investor will deposit all the available income in an account which will earn interest at 6% per annum effective, find the accumulated amount of the account in 22 years' time.

(b) Suppose that the bank loans may be repaid partially, but only at the end of each complete year, and that the investor may still deposit money at any time for any term at an annual rate of interest of 6% effective. Find

i. the discounted payback period for the project, and

ii. the accumulated amount in the investor's account in 22 years' time.

9. Let μ_x be the force of mortality, and ℓ_x the corresponding life table.

(i) Show that ${}_{n|m}q_x = \displaystyle\int_n^{n+m} {}_tp_x \mu_{x+t} dt$

[*Here* ${}_{n|m}q_x$ *means the probability of death of a life currently aged* x *between times* $x + n$ *and* $x + n + m$.]

(ii) Show that $\mu_{x+0.5} \approx -\log p_x$ and $\mu_x \approx -0.5 (\log p_x + \log p_{x-1})$.

(iii) If $\ell_x = 100(100 - x)^{1/2}$, find μ_{84} exactly.

10. (i) Gompertz's Law has $\mu_x^{(1)} = Bc^x$. Show that the corresponding survival function is given by ${}_tp_x^{(1)} = g^{c^x(c^t-1)}$ where $\log g = -B/\log c$.

(ii) Makeham's Law has $\mu_x^{(2)} = A + Bc^x$. Show that ${}_tp_x^{(2)} = s^t {}_tp_x^{(1)}$ where $s = e^{-A}$.

(iii) If $\mu_x = A \log x$, find an expression for ℓ_x/ℓ_0.

(iv) If it is assumed that A1967-70 table follows Makeham's Law, use ℓ_{30}, ℓ_{40}, ℓ_{50} and ℓ_{60} to find A, B and c.
($\ell_{30} = 33839$, $\ell_{40} = 33542$, and $\ell_{50} = 32670$, $\ell_{60} = 30040$).

11. The force of mortality for table 2 has twice the force of mortality for table 1.

(i) Show that the probability of survival for n years under table 2 is the square of that under table 1.

(ii) Suppose that table 1 follows the Gompertz Law from the previous question. Show that the probability of survival for n years for a life aged x under table 2 is the same as that under table 1 for a life aged $x + a$, for some $a > 0$. Find a. Comment on the result.

12. (a) Show that at age x if $0 \leq a < b \leq 1$ then ${}_{b-a}q_{x+a} = 1 - \dfrac{{}_bp_x}{{}_ap_x}$.

(b) Hence, or otherwise, show that if deaths occurring in the year of age $(x, x+1)$ are uniformly distributed that year, then ${}_{b-a}q_{x+a} = \dfrac{(b-a)q_x}{1 - aq_x}$.

Course webpage: `http://www.stats.ox.ac.uk/~winkel/sb4a.html`

4 Life insurance

Please hand in your answers to questions 1, 2(ii), 5, 7, 9, 10, 13, 16 and 19 for marking. The rest are for further practice.

1. (i) Show algebraically that $A_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$. Also demonstrate the result verbally.

   (ii) Show algebraically that $(IA)_x = \ddot{a}_x - d(I\ddot{a})_x$. Demonstrate the result verbally. [*Here* $(IA)_x$ *is the expected time-0 value (assuming an age of* x *at time 0) of a single payment of size* $K+1$ *made at time* $K+1$*, where death occurs in the year* $(K, K+1)$. $(I\ddot{a})_x$ *is the expected value of the stream of payments of 1 at time 0, 2 at time 1, ...,* $K+1$ *at time* K*, with the last payment made at the beginning of the year of death.*]

2. Suppose that the future lifetime of a life aged x ($x > 0$) is represented by a random variable T_x distributed on the interval $(0, \omega - x)$, where ω is some maximum age.

   (i) For $0 \leq x < y < \omega$, state a consistency condition between the distributions of T_x and of T_y.

   (ii) Suppose that $0 \leq x < y < \omega$, and let $t = y - x$. Define the force of mortality at age y: (a) in terms of T_0; and (b) in terms of T_x, and show that the two definitions are equivalent.

   (iii) Prove that ${}_tp_x = \exp\left\{ -\int_0^t \mu_{x+s} ds \right\}$.

3. Given a curtate lifetime $K = [T]$ and a constant-δ interest model, consider the following insurance products:

   (i) pure endowment with term n;

   (ii) whole life assurance;

   (iii) term assurance with term n;

   (iv) endowment assurance with term n (under this product, there is a payment of one unit at the end of the year of death or at end of term whichever is earlier).

   (a) Write each one as a random cash-flow of the form $C = \big((t_k, c_k B_k)\big)_{k=1,2,...}$, where the B_k are Bernoulli random variables defined in terms of K.

   (b) Find expressions for the net premiums and variances of these products.

   (c) Relate the products, premiums and variances of (iv) and (i) and (iii).

   (d) Comment on (a) and (b) for the corresponding products where payment is made at death rather than at the end of the year of death (and n is not necessarily an integer).

4. A cash-flow is payable continuously at a rate of $\rho(t)$ per annum at time t provided a life who is aged x at time 0 is still alive. T_x is a random variable which models the residual lifetime in years of a life aged x.

   (a) Write down an expression, in terms of T_x, for the (random) present value at time 0 of this cash-flow, at a constant force of interest δ p.a., and show that the expected present value at time 0 of the cash-flow is equal to

$$\int_0^\infty e^{-\delta s} \rho(s) \mathbb{P}(T_x > s) ds.$$

(b) An annuity is payable continuously during the lifetime of a life now aged 30, but for at most 10 years. The rate of payment at all times t during the first 5 years is £5,000 p.a., and thereafter £10,000 p.a. The force of mortality to which this life is subject is assumed to be 0.01 p.a. at all ages between 30 and 35, and 0.02 p.a. between 35 and 40. Find the expected present value of this annuity at a force of interest of 0.05 p.a.

(c) If the mortality and interest assumptions are as in (b), find the expected present value of the benefits of a term assurance, issued to the life in (ii), which pays £40,000 immediately on death within 10 years.

For the questions below, use the A 1967-70 Mortality table whenever table data is needed (all three columns, assuming medical checks were successful).

5. Find the present value of £5000 due in 5 years' time at $i = 4\%$ if

   a) the payment is certain to be made;

   b) the payment is contingent upon a life aged 35 now surviving to age 40.

6. How large a pure endowment, payable at age 65 can a life aged 60 buy with £1000 cash if $i = 7\%$?

7. a) Show that $\ddot{a}_x = 1 + (1+i)^{-1} p_x \ddot{a}_{x+1}$

   b) Show that $\ddot{a}_{x:\overline{n}|} = a_{x:\overline{n}|} + 1 - A_{\substack{1\\x:\overline{n}|}}$

   c) Show that $A_x = (1+i)^{-1}\ddot{a}_x - a_x$

8. Find the net single premium for a 5-year temporary life annuity issued to a life aged 65 if $i = 8\%$ for the first 3 years and $i = 6\%$ for the next 2 years.

9. Find the net annual premium for a £40000, 4-year term assurance policy issued to a life aged 26 if $i = 10\%$.

10. A deferred (temporary) life annuity is a deferred perpetuity (annuity-certain) restricted to a lifetime. For a life aged x, deferred period of m years (and a term of n years, from $m+1$ to $m+n$), the notation for the single premium is ${}_{m|}a_x$ (respectively ${}_{m|}a_{x:\overline{n}|}$).

    a) Give expressions for ${}_{m|}a_x$ and ${}_{m|}a_{x:\overline{n}|}$ in terms of the life table probabilities q_k.

    b) Show that ${}_{m|}a_x = A_{\substack{1\\x:\overline{m}|}} a_{x+m}$.

11. Describe the benefit which has the present value random variable function given by Z below; T denotes the future lifetime of a life aged x.

$$Z = \begin{cases} \bar{a}_{\overline{n}|} & T \le n \\ \bar{a}_{\overline{T}|} & T > n \end{cases}$$

12. A special deferred annuity provides as benefits for a life aged 60:

    - on survival to age 65 an annuity of £2,000 p.a. payable in advance for two years certain and for life thereafter

    - on death between ages 63 and 65, £5,000 payable at the end of the year of death

    - on death between ages 60 and 63, £10,000 payable at the end of the year of death.

Annual premiums are payable in advance until age 65 or earlier death. Determine the level annual premium based on an effective interest rate of 4% p.a. For the life annuity part, you may approximate all one-year death probabilities for age greater than 65 by the constant 0.05 (which leads close to the correct numerical answer).

13. Give an expression (in terms of standard actuarial functions) for the annual premium for a 25 year endowment assurance on a life aged 40. The initial expenses are £2 per £100 sum insured, renewal expenses are 5% of each premium and 30p each year per £100 sum insured, and there is an initial commission of 50% of the first year's gross premium.

14. Some time ago, a life office issued an assurance policy to a life now aged exactly 55. Premiums are payable annually in advance, and death benefits are paid at the end of the year of death. The office calculates reserves using gross premium policy values. The following information gives the reserve assumptions for the policy year just completed. Expenses are assumed to be incurred at the start of the policy year.

    Reserve brought forward at the start of the policy year: £12,500

    Annual premium: £1,150

    Annual expenses: £75

    Death benefit: £50,000

    Mortality: A1967/70

    Interest 5.5% per annum

    Calculate the reserve at the end of the policy year.

15. A deferred annuity is purchased by 20 annual payments payable by a life aged 40 for a year annuity in advance of £2,500 a year, commencing in 20 years, for life. Find an expression for the premium on the basis of 4% pa interest with expenses of 5% of each premium and £5 at each annuity payment.

16. Suppose that $l_x = 100,000(100 - x)$, where $0 \le x \le 100$, and the interest rate is 5%.

    (a) Calculate $A_{50:\overline{10}|}$ and $\ddot{a}_{50:\overline{10}|}$.

    (b) Calculate the net annual premium for a 10 year endowment assurance for £10,000 to someone aged 50 and the policy values of years 3 and 4 using the values above.

    (c) Suppose that expenses are as follows

Commission:	50% of First Premium
	2% of Subsequent Premiums
General Expenses:	£150 Initially
	£10 in each subsequent year.

    Calculate the office premium for the policy in (b).

17. Show algebraically that the product of

    - the reserve after t years for an annual premium n-year pure endowment issued to a life aged x

    and

- the annual premium for pure endowment of like amount issued at the same age but maturing in t years

is constant for all values of t. (Ignore expenses.)

Try to find an argument for this by general reasoning also.

18. (i) Show that $\overline{A}_x = 1 - \delta\overline{a}_x$.

    (ii) Consider a whole life assurance with sum assured 1 payable at the point of death, with a constant premium paid continuously. Show that the reserve at time t satisfies

$$ {}_t\overline{V}_x = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x} $$

19. Thiele's equation for the policy value at duration t, ${}_t\overline{V}_x$, of a life annuity payable continuously at rate 1 from age x is:

$$ \frac{\partial}{\partial t} {}_t\overline{V}_x = \mu_{x+t}\ {}_t\overline{V}_x - 1 + \delta\ {}_t\overline{V}_x. $$

Derive this result algebraically, and also explain it by general reasoning.

20. Let ${}_tV_x$ be the reserve at time t on an insurance policy issued to a life aged x that pays a benefit of $S_x(t)$ on death at time t, under which premiums are payable of $P(t)$ at time t. Show by general reasoning that

$$ \frac{\partial}{\partial t} {}_tV_x = P(t) + \delta\ {}_tV_x - \mu_{x+t}(S_x(t) - {}_tV_x) $$

where δ is the force of interest.

21. An insurer issues n identical policies. Let Y_j be the claim amount from the jth policy, and suppose that the random variables Y_j, $j = 1, \ldots, n$ are i.i.d. with mean $\mu > 0$ and variance σ^2. The insurer charges a premium of A for each policy.

    (a) Show that if $A = \mu + 10\sigma n^{-1/2}$, then the probability that total claims exceed total premiums is no more than 1%, for any value of n.

    (b) Use the Central Limit Theorem to show that if instead $A = \mu + 3\sigma n^{-1/2}$, then this probability is still less than 1%, provided n is large enough.